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GLOBAL DYNAMICS OF NEWTON'S METHOD FOR COMPLEX POLYNOMIALS

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Abstract

Newton's method, as a root-finding algorithm, has been used since ancient times to solve daily problems. Nevertheless, it was not until the second half of the nineteenth century that it began being studied as a dynamical system in the complex plane. Following this path, the main goal of this thesis is to understand and prove, using recently developed techniques, Shishikura's result on the connectivity of the Julia set of the Newton map of polynomials. To do so, we first present a set of preliminary tools that contain normal families, conformal representations and proper maps, among others. It is followed by a study of rational complex dynamical systems, some results on the existence of fixed points of meromorphic maps and it is concluded by what is the cornerstone of this project: the proof of the connectivity of the Julia set of Newton maps of polynomials.

El mètode de Newton, com a algorisme per a trobar arrels de funcions, s'ha utilitzat des de temps ancestrals per a solucionar problemes quotidians. Tanmateix, no va ser fins a la segona meitat del segle dinou que va començar a estudiar-se des del punt de vista dels sistemes dinàmics complexos. Seguint aquest estudi, l'objectiu principal d'aquest informe és entendre i demostrar, tot utilitzant eines desenvolupades recentment, el resultat obtingut per Shishikura sobre la connectivitat del conjunt de Julia de funcions de Newton per a polinomis. Per a fer-ho, en primer lloc presentem un seguit de resultats preliminars, entre els quals hi ha resultats sobre famílies normals, representacions conformes i funcions pròpies, entre d'altres. Seguidament es realitza un estudi sobre sistemes dinàmics complexos de funcions racionals, es presenten resultats sobre l'existència de punts fixos de funcions meromorfes i es conclou el treball amb el que és la peça clau: la demostració de la connectivitat del conjunt de Julia de funcions de Newton per a polinomis.

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Contents

Introduction

Since ancient times, humanity has tried to understand and model the world. In order to do so, we developed the mathematical language, and from then on logical problems usually required solving one or more equations. One of the oldest and probably most famous methods to numerically solve an equation is Newton's method. This technique, first introduced by [1]Sir Isaac Newton in 1669 and later refined by [2]Joseph Raphson and [3]Thomas Simpson, was originally conceived to approximate real (and later on, complex) solutions to the equation $f(z) = 0$. It is also an iterative method, meaning that at each step, the value of the previous result is needed in order to compute the new value. Choosing an initial point z_0, sometimes called seed, near the desired solution, Newton's method presents an n^{th} approximation of the root using the formula

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}.$$

Although originally proposed to find real solutions to equations, [4]Ernst Schröder and [5]Arthur Cayley were the first to study this method in the complex plane. The German mathematician and logician Ernst Schröder published a two-part paper in 1870 and 1871 where he studied the iteration of rational functions (which are the quotient of two polynomials) in the complex plane and stated his fixed-point theorem, whereas the British mathematician Arthur Cayley is often credited with the first proof of this result despite being found in 1879.

A wider problem they were interested in was the behaviour of the iterates of Newton's method far from the roots, this is, to determine which seeds in the complex plane produce a sequence of iterates converging to a root α of $f(z)$. Consider the *Newton map* of a function f, defined as

$$N(z) = z - \frac{f(z)}{f'(z)}.$$

One can show that the only finite fixed points of N are no other than the zeroes of f. Furthermore, if α is a zero of f, a sequence of iterates of a point close enough to α under N converges to α, this is, all roots of f are attracting fixed points of N. Generally, if α is an attracting fixed point of an arbitrary map F, the *basin of attraction* of α is defined as the set of points in the complex plane that produce a sequence of iterates converging to α. This is,

$$A(\alpha) = \{z \in \mathbb{C} \mid F^n(z) \underset{n\to\infty}{\longrightarrow} \alpha\},$$

[1]Sir Isaac Newton. English mathematician, physicist, astronomer and theologian, 1642 - 1726.
[2]Joseph Raphson. English mathematician, 1648 - 1715.
[3]Thomas Simpson. English mathematician and inventor, 1710 - 1761.
[4]Friedrich Wilhelm Karl Ernst Schröder. German mathematician and logician, 1841 - 1902.
[5]Arthur Cayley. English mathematician, 1821 - 1895.

where $F^n(z)$ is the composition $F \circ \overset{n}{\cdots} \circ F\ (z)$. So, going back to $N(z)$, they were essentially interested in the classification of points on the complex plane in the different basins of attraction of N. Both Schröder and Cayley solved this problem for quadratic polynomials, showing that both basins are a half-plane, but they could not obtain a similar result for higher degree polynomials.

The next big step in the study of global theory was not taken until the beginning of the twentieth century, when French mathematicians [6]Pierre Fatou and [7]Gaston Julia studied in a systematic way the iteration of rational functions in the Riemann sphere. Although Fatou already made some discoveries before, it was in 1915 when the French Academy of Sciences, motivated by [8]Henri Poincaré's use of iteration in his studies of celestial mechanics, offered the 1918 *Grand Prix des Sciences mathématiques* for the study of iteration. Both Fatou and Julia made brilliant breakthroughs between 1917 and 1919, but it was Julia who, with a magnificent work, went away with the award, since Fatou did not participate in the prize probably due to a late submission of very similar results to those of Julia. It is said that Julia publicly accused Fatou of stealing his results, although it was never proven.

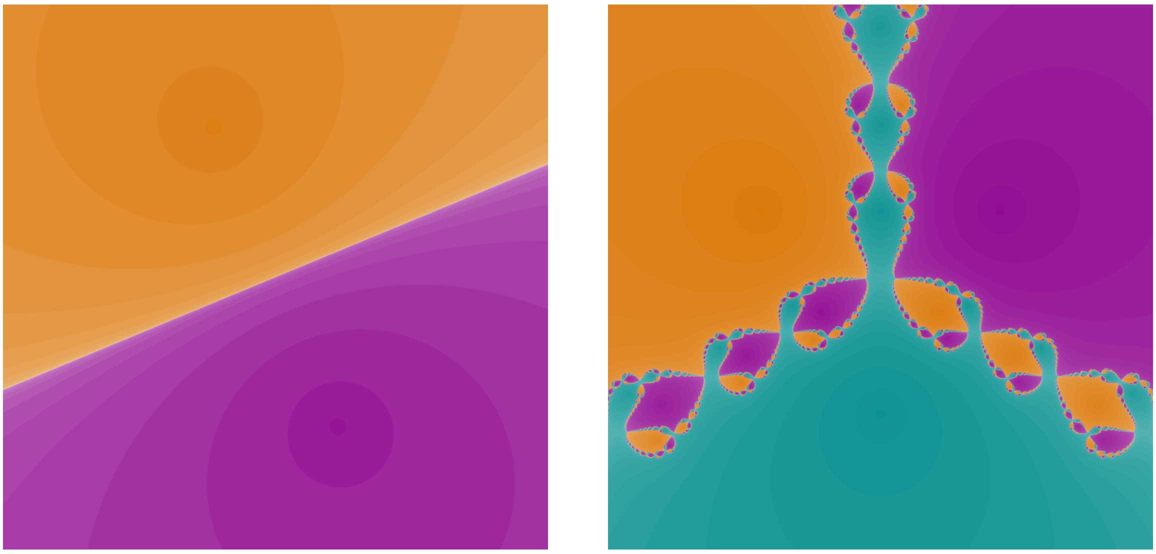
The most pioneering technique introduced by these two gifted mathematicians was the use of normal families to divide the Riemann sphere into two totally invariant regions, later named after them. Naively, given a complex map F one can think of the Julia set, $\mathcal{J}(F)$, as the points whose behaviour is very different from that of their neighbours under iteration of F. Conversely, the Fatou set, $\mathcal{F}(F)$, is the complement of the Julia set, and consists of points that behave very similarly to their neighbours under iteration of F. These informal definitions justify why the Fatou set is often called the stable set, and the Julia set is known as the unstable or chaotic set. Figure 1 presents two dynamical planes of Newton maps of polynomials of degree two and three, in which one can see that the Julia set (in white, separating the different basins of attraction) is much more intricate in the cubic case than in the quadratic one.

Nonetheless, these astonishing pictures could only be imagined by Fatou and Julia, and it was not until the raise of modern computers, around 1980's, that mathematicians could gaze this marvellous images. One of the first mathematicians to see such a breathtaking fractal was [9]Benoit Mandelbrot in 1980, while working at an IBM research centre. The set he printed, which was later named after him, is the well known Mandelbrot set. In order to define it, we must first consider the quadratic family, $\{Q_c(z) = z^2 + c\}_{c\in\mathbb{C}}$, and the filled Julia set $\mathcal{K}_c$, which is the set of points whose iterates under Q_c do not diverge to infinity. Then, the Mandelbrot set is defined as

$$\mathcal{M} = \{c \in \mathbb{C} \mid \mathcal{K}_c \text{ is connected}\}.$$

This field, complex dynamics, is related to many other branches in mathematics and these fruitful relations led to numerous awards to remarkable mathematicians, such as the Fields Medals awarded to [10]J. C. Yoccoz (1994) and to C. T. McMullen (1998). Nevertheless, it is still a field full of conjectures and open problems, like the MLC conjecture, which wonders if the Mandelbrot set is locally connected. The fact that still nowadays there are lots of unanswered

[6]Pierre Joseph Louis Fatou. French mathematician and astronomer, 1878 - 1929.
[7]Gaston Maurice Julia. French mathematician, 1893 - 1978.
[8]Jules Henri Poincaré. French mathematician, theoretical physicist and engineer, 1854 - 1912.
[9]Benoit B. Mandelbrot. Polish-born French and American mathematician, 1924 - 2010.
[10]Jean-Christophe Yoccoz. French mathematician, 1957 - 2016.



(a) Newton's method for a quadratic polynomial. (b) Newton's method for a cubic polynomial.

Figure 1: Dynamical planes of two Newton maps of polynomials. In (a) we see the Newton map associated with the quadratic polynomial $P(z) = z^2 + 1 + i$, which has only two roots. The Julia set (in white) separates the two basins of attraction of each root. In (b) is presented the Newton map associated with the cubic polynomial $P(z) = (z - i)(z - a)(z + \overline{a})$ with $a = 0.8763 + 4818i$, which has three distinct roots. The Julia set (in white, separating the different basins of attraction) is much more complicated than in the former case.

questions is a great stimulus that has drawn the attention of exceptional minds, leading to recent breakthroughs in the field.

One of these open questions which has been considered in the literature is the connectivity of the Julia set of Newton's maps. It is worth noting that, since the Julia set is compact in the Riemann sphere, its connectivity is equivalent to the simple connectivity of all Fatou components (i.e., the connected components of the Fatou set). The main goal of this work is to understand and prove with the utmost rigour the following theorem regarding this matter, first proven in 1989 [Shi1] by Mitsuhiro Shishikura and later simplified in 2009 [Shi2].

Theorem A. *Let P be a polynomial and $N = z - \frac{P(z)}{P'(z)}$ its Newton map. Then all Fatou components of N are simply connected.*

Shishikura proved a more general result using quasiconformal surgery. He proved that every rational map R with less than two weakly repelling points (i.e., fixed points $R(z_0) = z_0$ such that $|R'(z_0)| > 1$ or $R'(z_0) = 1$), like any Newton map of a polynomial, has a connected Julia set. Our goal is not to mimic Shishikura's proof, which was done by studying pullbacks of invariant absorbing sets under N, but instead to use a recently developed technique by Barański *et al.* [BFJK1] to prove the stated result. In order to do so, we will also prove a second result, used widely in the proof of the previous.

Theorem B. *Let P be a polynomial, $N = z - \frac{P(z)}{P'(z)}$ its Newton map, α a root of P and A the immediate basin of attraction of α. Then, A is unbounded.*

This statement is not true for a general rational map since, for example, the rational map $Q_0(z) = z^2$ has a bounded immediate basin of attraction $A(0)$.

Although it was Shishikura who proved the result for a Newton map of an arbitrary polynomial, other authors had previously made advancements in this direction. In 1989 and previous to Shishikura's proof, Przytycki [Prz] proved that every root of P has a simply connected immediate basin as a fixed point of N_P. Also in 1989, Meier [Mei] proved, by a completely different method, the connectivity of $\mathcal{J}(N_P)$ for degree 3 polynomials, and later in 1997 [11]Tan Lei [Tan] generalized Meier's result for higher degree polynomials.

Once the connectivity of the Julia set was proven for Newton maps of polynomials, the technique was brought to prove the same result for entire transcendental maps, but it was not an easy task. Nevertheless, making use of extra tools, the strategy worked for all Fatou components except for Baker domains (a type of Fatou component that does not exist for rational maps) as proved in [BT, FJT1, FJT2]. Finally, in 2014, Barański *et al.* [BFJK1] proved the connectivity for Baker domains, hence concluding that the Julia set of the Newton map of an entire transcendental map is indeed connected.

It is difficult to find direct applications of the abstract results presented in this work, and they are often only studied for its intrinsic beauty. Nonetheless, Hubbard *et al.* [HSS] found a technique that, given a polynomial of degree d, provides a seed in every basin of attraction of its Newton map. This means that iterating these seeds under N, the technique produces every possible root of the polynomial. They claim that the cardinality of such a set of seeds can be as small as $1.11d\log^2 d$, but it can be reduced to $1.30d$ if all the roots are real. Furthermore, they not only show the previous statement, but they also provide a recipe to construct this set, whose elements lie in equally spaced circles inside a round annulus centred at the origin.

The aim of this work is to prove the result obtained by Shishikura in 1989 but using techniques developed by Barański *et al.* in [BFJK1] for Baker domains, which do not require quasiconformal surgery. This strategy is explained in general in [BFJK2], and we adapt it here to the polynomial case.

In order to do so, we devote the first chapter to present important tools and preliminary results that are very useful in later proofs. Amongst them, normal families, some results on conformal representations, like distortion theorems, and proper maps.

In Chapter 2 our aim is to understand and present results on complex dynamical systems, particularly of rational maps. We first define and establish the basic tools of dynamical systems, before focusing on describing periodic points and orbits, and how to classify them by its nature. We then step into the world of rational maps, and characterize its critical points and values, which play an important role. We end this chapter by formally defining the Fatou and Julia sets, along with some useful and essential properties, and stating the Classification Theorem of Fatou components.

Chapter 3 is dedicated to build some results which share the aim of finding fixed points of meromorphic maps. We first briefly introduce the notion of polynomial-like and rational-like mappings, and then develop some tools that allow us to prove the existence of fixed points of meromorphic maps under some hypotheses.

All these results give way to Chapter 4, where we formally define the Newton map of a polynomial and give some fundamental properties about it. Following this, we prove what

[11]Tan Lei. Chinese mathematician, 1963 - 2016.

is the keystone of this thesis, both Theorem A and Theorem B. We prove Theorem A by supposing there are multiply connected Fatou components and then finding weakly repelling points, which a Newton map does not have. Theorem B is proved by assuming the immediate basin of attraction of a fixed point of N_P is bounded, and then finding a fixed point in its boundary, leading to a contradiction.

Finally, two appendices are presented. The first gathers a collection of background results, used throughout the project and learned by the author during his bachelor degree. The second provides a link where one can find all the code developed to obtain the images in this thesis, as well as some high-resolution images.

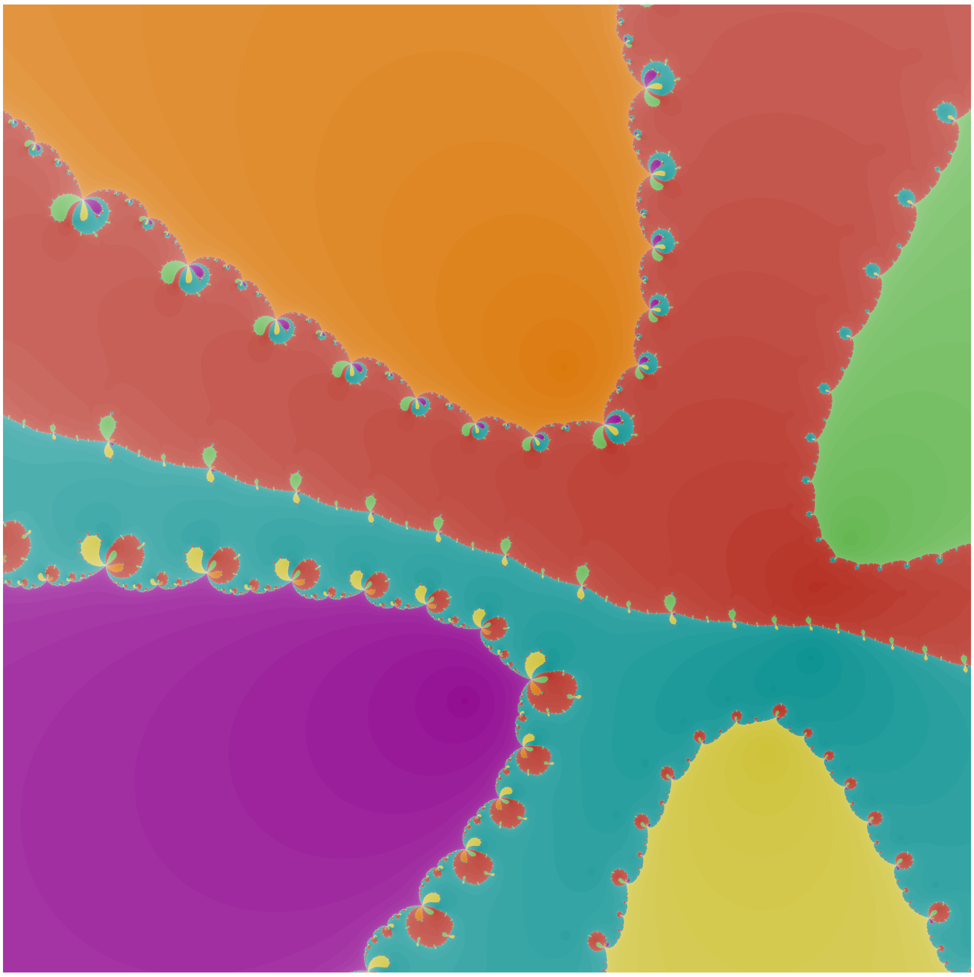


Figure 2: Dynamical plane of a Newton map of a polynomial of degree 6. Every immediate basin of attraction seems to be unbounded, and every Fatou component seems to be simply connected. These claims will be proven throughout this thesis.

Chapter 1

Preliminary results

The main purpose of this chapter is to introduce some useful results that will be used in sections later on. Most of them can be found in [Ahl, CG, Con, McM, MH, Mil, Pom, Ste] or similar complex analysis and complex dynamics books.

To begin with, we introduce the notion of normal family of functions, an essential concept in dynamical systems. It is followed by some results on conformal representations, which include distortion theorems, and it is concluded by utile facts about proper maps.

While working on the complex plane with rational functions, it is useful to extend the maps to the Riemann sphere, considering infinity as an ordinary point.

Definition 1.1. (Complex sphere) We define the *extended complex plane*, the *Riemann sphere* or the *complex sphere* as the union $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. To obtain a metric on $\widehat{\mathbb{C}}$ we identify $\mathbb{C}$ with the horizontal plane $\mathbb{R}^2$, and consider the stereographic projection $\pi : \widehat{\mathbb{C}} \to S^2$, which is a bijection between the plane and the unit sphere S^2. Then we are able to define the *chordal metric*, which is the distance between two points in the sphere:

$$\sigma(z_1, z_2) = |\pi(z_1) - \pi(z_2)| = \frac{2\,|z_1 - z_2|}{(1 + |z_1|^2)^{1/2}(1 + |z_2|^2)^{1/2}}, \ \forall z_1, z_2 \in \widehat{\mathbb{C}}\ .$$

Now, the distance between a point z and ∞ is simply

$$\sigma(z, \infty) = \lim_{z_0 \to \infty} \sigma(z, z_0) = \frac{2}{(1 + |z_1|^2)^{1/2}}, \ \forall z \in \widehat{\mathbb{C}}\ .$$

It is convenient to introduce the following notation, used from now on.

Notation. A disk in the complex plane centred at z_0 and of radius r will be denoted by

$$\mathbb{D}(z_0, r) := \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

The unit disk and the centred disk will be denoted respectively by

$$\mathbb{D} := \mathbb{D}(0, 1) \quad \text{and} \quad \mathbb{D}_r := \mathbb{D}(0, r).$$

1.1 Normal families

Definition 1.2. A sequence $\{f_n\}_n$ of maps from a metric space (X, d_1) to (Y, d_2) *converges locally uniformly* on $U \subseteq X$ to some map f if each point $x \in U$ has a neighbourhood on which f_n converges uniformly to f.

Definition 1.3. (Normal family) A family $\mathscr{F} = \{f_n : X \to \widehat{\mathbb{C}}\}_n$ of holomorphic maps is said to be a *normal family* in an open set U if every infinite sequence of functions from $\mathscr{F}$ contains a subsequence which converges locally uniformly on U (equivalently, that converges uniformly on compact subsets of U), i.e., if $\forall f_{n_k}, \exists f_{n_{k_j}}$ such that $f_{n_{k_j}|U} \rightrightarrows f$. The limit is again a holomorphic map to $\widehat{\mathbb{C}}$.

Theorem 1.4. (Montel's Theorem, [CG, pp. 10-11] or [Bea, §3.3]**)** *Given a family of holomorphic maps $\mathscr{F} = \{f_n : U \to \widehat{\mathbb{C}}\}_n$, if they all omit at least 3 different values in $\widehat{\mathbb{C}}$, i.e., if exist three different values $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ such that $\bigcup_n f_n(U) \subset \widehat{\mathbb{C}} \setminus \{z_1, z_2, z_3\}$, then $\mathscr{F}$ is a normal family in U.*

It is sometimes assumed in the literature that $\{z_1, z_2, z_3\} = \{0, 1, \infty\}$. This is because one can compose every f in the family with a Möbius transformation g (which is conformal in $\widehat{\mathbb{C}}$) fulfilling $g(z_1) = 0$, $g(z_2) = 1$ and $g(z_3) = \infty$.

1.2 Conformal representations

The following result is one of the fundamental theorems in complex analysis, and although first formulated by [12]B. Riemann in 1851 in his Ph.D. thesis, its first successful proof was due to [13]C. Carathéodory in 1912, who used Riemann surfaces. Two years later [14]P. Koebe simplified it.

Theorem 1.5. (Riemann's mapping Theorem, [Ahl, pp. 230-231]**)** *Let $U \in \mathbb{C}$ be a non-empty simply connected open set, $U \neq \mathbb{C}$. Then exists a conformal map φ (called* Riemann mapping*) from U onto the open unit disk $\mathbb{D}$. Given $z_0 \in U$ there is exactly one Riemann mapping with $\varphi(z_0) = 0$, up to rotation.*

Theorem 1.6. (Carathéodory's Theorem, [Pom, p. 24]**)** *Let $U \in \mathbb{C}$ be a non-empty simply connected open set, $U \neq \mathbb{C}$, and $\varphi : U \to \mathbb{D}$ a Riemann mapping. If the boundary of U is locally connected then the Riemann mapping extends continuously to the boundary, $\widetilde{\varphi}^{-1} : \partial\mathbb{D} \to \partial U$.*

The following distortion theorems will be used in the proof of Theorem B.

Theorem 1.7. (Koebe's distortion Theorem, [McM, pp. 15-16]**)** *Let $\Phi : \mathbb{D}(a, 1) \to \mathbb{C}$ be a univalent map and $0 < r < 1$. Then exists a constant $C(r) \geq 1$ such that $\lim_{r \to 0} C(r) = 1$ and for all $x, y \in \mathbb{D}(a, r)$*

$$\frac{1}{C(r)} \left| \Phi'(a) \right| \leq \frac{|\Phi(x) - \Phi(y)|}{|x - y|} \leq C(r) \left| \Phi'(a) \right|.$$

[12]Georg Friedrich Bernhard Riemann. German mathematician, 1826 - 1866.
[13]Constantin Carathéodory. Greek mathematician, 1873 - 1950.
[14]Paul Koebe. German mathematician 1882 - 1945.

Remark 1.8. From the Koebe distortion Theorem, we can conclude:

(1) Taking the limit $y \to x$ we have $\frac{1}{C(r)}\Phi'(a) \leq |\Phi'(x)| \leq C(r)\Phi'(a)$, $\forall x \in \mathbb{D}(a,r)$.

(2) Particularly, we have $\frac{1}{C(r)^2} \leq \left|\frac{\Phi'(x)}{\Phi'(y)}\right| \leq C(r)^2$, $\forall x, y \in \mathbb{D}(a,r)$, which is known as *bounded distortion*.

(3) Observe that x and y need to belong to a smaller disk than the maximum domain of conformality, and the smaller the disk, the tighter the bounds are.

(4) Observe that $C(r)$ is independent of Φ, and only depends on r.

Definition 1.9. ($n-$connectedness) An open set $U \subset \mathbb{C}$ is *$n-$connected* if and only if the number of connected components of its boundary is exactly n.

Definition 1.10. (Annulus) An *annulus* is an open region of space which is 2-connected, i.e., whose fundamental group is isomorphic to $\mathbb{Z}$. It can be seen as the region in between two topological disks, U and U' with $\overline{U'} \subset U$, that is, $A = U \setminus U'$. A *round annulus* is an annulus bounded by concentric disks, i.e.,

$$\mathbb{A}(z_0, r, R) := \{z \in \mathbb{C} \mid r < |z - z_0| < R\}.$$

If the annulus is centred at the origin and has outer radius one we will call it a *standard annulus*, $\mathbb{A}_r := \mathbb{A}(0, r, 1) = \{z \in \mathbb{C} \mid r < |z| < 1\}$.

Definition 1.11. (Modulus of an annulus) Given an annulus A there is a unique (up to rotation) conformal map φ such that $\varphi(A) = \mathbb{A}_r$. This number r is a conformal invariant (i.e. is preserved under conformal maps) and allows us to define the *modulus* of A:

$$\mathrm{mod}(A) = \mathrm{mod}(\mathbb{A}_r) := -\frac{\log r}{2\pi}.$$

Example 1.12. Consider the punctured disk $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$. In this case the modulus is defined by continuity to be $\mathrm{mod}(\mathbb{D}^*) = \infty$. Another illustrative example would be the degenerate case, the annulus $\mathbb{A}_r$ when $r \to 1$. In this scenario, $\mathrm{mod}(\mathbb{A}_r) \underset{r\to 1}{\longrightarrow} 0$. It is clear now that $\mathrm{mod}(A) \in (0, \infty]$, for any annulus A.

Knowing these previous definitions we are able to rewrite Theorem 1.7 in terms of the modulus of an annulus.

Theorem 1.13. *Let U, U' be two topological disks in $\mathbb{C}$ such that $\overline{U'} \subset U$, $m = \mathrm{mod}(U \setminus U')$ and $\Phi : U \to \mathbb{C}$ a conformal map. Then exists a constant $C(m) > 0$ such that for any x, y and z in U',*

$$\frac{1}{C(m)} \left|\Phi'(x)\right| \leq \frac{|\Phi(y) - \Phi(z)|}{|y - z|} \leq C(m) \left|\Phi'(x)\right|.$$

Moreover, $\lim\limits_{m\to\infty} C(m) = 1$.

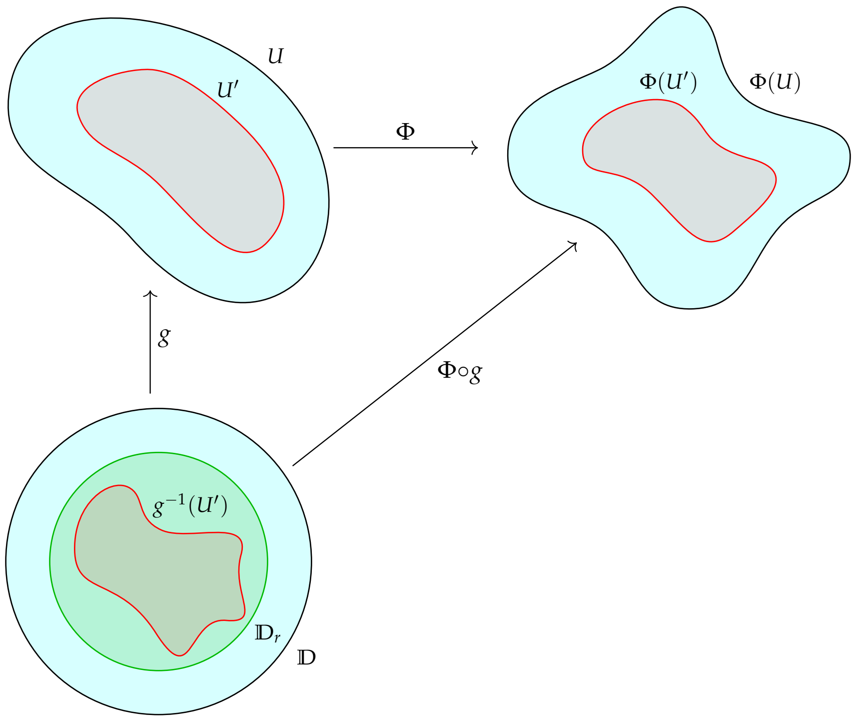


Figure 1.1: Diagram of conformal maps between $\mathbb{D}$, U and $\Phi(U)$ used in the proof of Theorem 1.13

Proof. Since U is a non-empty simply connected open set, there is a Riemann mapping from U onto $\mathbb{D}$. Denote the inverse of such a conformal map by $g : \mathbb{D} \to U$. Since $\overline{U'} \subset U$, we know there is a constant $r(m) < 1$ that depends only on the modulus m such that $g^{-1}(U') \subset \mathbb{D}_r \subset \mathbb{D}$ (see Figure 1.1). Now we apply Koebe's distortion Theorem to both g and $\Phi \circ g$, where $a = 0$ is the centre of the disk and $\widetilde{x}, \widetilde{y} \in g^{-1}(U') \subset \mathbb{D}_r$, obtaining the following inequalities:

$$\frac{1}{\widetilde{C}(r)} \left|g'(0)\right| \leq \frac{|g(\widetilde{x}) - g(\widetilde{y})|}{|\widetilde{x} - \widetilde{y}|} \leq \widetilde{C}(r) \left|g'(0)\right|$$
$$\frac{1}{\widetilde{C}(r)} \left|\Phi'(g(0))\right| \left|g'(0)\right| \leq \frac{|\Phi(g(\widetilde{x})) - \Phi(g(\widetilde{y}))|}{|\widetilde{x} - \widetilde{y}|} \leq \widetilde{C}(r) \left|\Phi'(g(0))\right| \left|g'(0)\right|$$

Combining both expressions, we have

$$\frac{1}{\widetilde{C}(r)^2} \left|\Phi'(g(0))\right| \leq \frac{|\Phi(g(\widetilde{x})) - \Phi(g(\widetilde{y}))|}{|g(\widetilde{x}) - g(\widetilde{y})|} \leq \widetilde{C}(r)^2 \left|\Phi'(g(0))\right|.$$

Let $x := g(0)$, $y := g(\widetilde{x})$, and $z := g(\widetilde{y})$, where $x, y, z \in U'$, and note that x could be any point in U' because g is the inverse of a Riemann mapping. Defining $C(m) := \widetilde{C}(r(m))^2$, we get

$$\frac{1}{C(m)} \left|\Phi'(x)\right| \leq \frac{|\Phi(y) - \Phi(z)|}{|y - z|} \leq C(m) \left|\Phi'(x)\right|,$$

which proves the result.

Additionally, since $m \to \infty$ implies $r \to 0$, and by Koebe's we know that $\lim_{r\to 0} \widetilde{C}(r) = 1$, then

$$\lim_{m\to\infty} C(m) = \lim_{r\to 0} \widetilde{C}(r)^2 = 1.$$

$\square$

A consequence of the theorem above, often used in complex dynamics and which will be used in the proof of Theorem B, is the following.

Theorem 1.14. *Let U, U' be two topological disks in $\mathbb{C}$ such that $\overline{U'} \subset U$, $a \in U'$ and $\Phi : U \to \mathbb{C}$ a conformal map. Let*

$$R_{max} := \max_{z \in \partial U'} |\Phi(z) - \Phi(a)|$$
$$R_{min} := \operatorname{dist}(\Phi(a), \partial\Phi(U')) = \min_{z \in \partial U'} |\Phi(z) - \Phi(a)|\,,$$

as shown in Figure 1.2. Then there is a constant k independent of Φ such that

$$1 \leq \frac{R_{max}}{R_{min}} \leq k.$$

In particular,

$$\mathbb{D}(\Phi(a),\ \frac{1}{2k} \cdot \operatorname{diam} \Phi(U')) \subset \Phi(U').$$

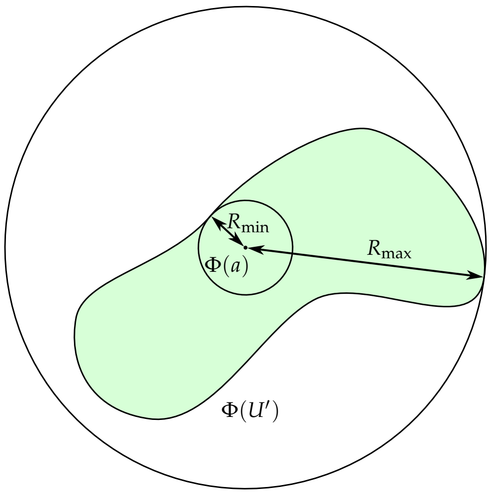


Figure 1.2: Geometric interpretation of distances $R_{\min}$ and $R_{\max}$ respect to $\Phi(U')$.

Proof. Since $\overline{U'} \subset U$, and both U and U' are open and simply connected we can find another open simply connected set U'' such that $\overline{U'} \subset U''$ and $\overline{U''} \subset U$.

Denoting $m' = \operatorname{mod}(U \setminus U'')$, we are under the same hypotheses as Theorem 1.13, and considering $x = y = a \in U' \subset U''$, we have

$$\frac{1}{C(m')} \left|\Phi'(a)\right| \leq \frac{|\Phi(z) - \Phi(a)|}{|z - a|} \leq C(m') \left|\Phi'(a)\right|, \quad \forall z \in U'',\ z \neq a.$$

Multiplying everywhere by $|z - a|$:

$$\frac{1}{C(m')} \left|\Phi'(a)\right| |z - a| \leq |\Phi(z) - \Phi(a)| \leq C(m') \left|\Phi'(a)\right| |z - a|\,.$$

Now, since $z \in \partial U' \subset U''$, from the last inequalities we can bound $R_{\max}$ and $R_{\min}$:

$$R_{\max} \leq C(m') \left|\Phi'(a)\right| \max_{z \in \partial U'} |z - a|$$
$$R_{\min} \geq \frac{1}{C(m')} \left|\Phi'(a)\right| \min_{z \in \partial U'} |z - a| = \frac{1}{C(m')} \left|\Phi'(a)\right| \operatorname{dist}(a, \partial U'),$$

and dividing these last two we obtain

$$1 \leq \frac{R_{\max}}{R_{\min}} \leq \frac{C(m')\,|\Phi'(a)|\max_{z\in\partial U'}|z-a|}{\frac{1}{C(m')}\,|\Phi'(a)|\,\mathrm{dist}(a,\partial U')} = \frac{C(m')^2\max_{z\in\partial U'}|z-a|}{\mathrm{dist}(a,\partial U')} =: k.$$

Therefore, since $2R_{\max} \geq \mathrm{diam}(\Phi(U'))$ $^{(*)}$, we have

$$R_{\min} \geq \frac{R_{\max}}{k} \underset{(*)}{\geq} \frac{\mathrm{diam}(\Phi(U'))}{2k}.$$

This implies that, for every $a \in U'$, we can find a disk centred at $\Phi(a)$ with a radius comparable to the diameter of $\Phi(U')$ entirely inside $\Phi(U')$. □

Remark 1.15. Observe that this result asserts that inside $\Phi(U')$ we can define a disk centred at $\Phi(a) \in \Phi(U')$ with a radius proportional to the diameter of $\Phi(U')$, with the constant of proportionality being independent of Φ. Given a family of conformal functions $\Phi_n : U \to \mathbb{C}$, and $\overline{U'} \subset U$, this result claims that that the sets $\Phi_n(U')$ cannot become sausage-shape, and that if they are shrinking they cannot converge to a segment but need to converge to a point.

1.3 Proper maps

Rational functions are proper of the Riemann sphere, but they are also proper when restricted to certain sets. Although properness is a topological concept, we go over some useful properties which occur in the holomorphic setting.

Definition 1.16. (Proper map) Let S, T be two topological spaces, and $f : S \to T$ a continuous map. Then, f is *proper* if, for any compact set $K \subset T$, its full preimage $f^{-1}(K)$ is compact in S. For every point $y \in T$ its full preimage $f^{-1}(y)$ is called *fiber* of f over y.

Remark 1.17. Discrete fibers are finite: let $f : S \to T$ be a proper map and consider $y \in T$, which is compact. Therefore $f^{-1}(y)$ is compact in S. If the number of components of the fiber $f^{-1}(y)$ were to be infinite the fiber should have an accumulation point and thus it would not be discrete.

Example 1.18. Consider the map $f(z) = z^2$ defined in U onto V, as shown in Figure 1.3. The set K is a compact set in V, but its preimage $f^{-1}(K)$ is not compact in U.

Proposition 1.19. *Let $U, V \subset \widehat{\mathbb{C}}$, and $f : U \to V$ a non-constant holomorphic map continuously well defined in ∂U. Then, f is proper if and only if $f(\partial U) = \partial V$.*

In order to prove the former proposition we need the following corollary of the Open Mapping Theorem (Theorem A.9).

Corollary 1.20. *Let $f : U \to V$ be a non-constant holomorphic map continuously well defined in ∂U. Then $\partial V \subset f(\partial U)$.*

Proof. Let $y \in \partial V$, and suppose that at least one preimage $f^{-1}(y)$ is in $\operatorname{int} U$. Then there is an open neighbourhood W of $f^{-1}(y)$ which is contained entirely in U. Since f is non-constant and holomorphic, by the Open Mapping Theorem f is open, and then $f(W)$ is an open neighbourhood in V which contains y, leading to a contradiction since $y \in \partial V$. □

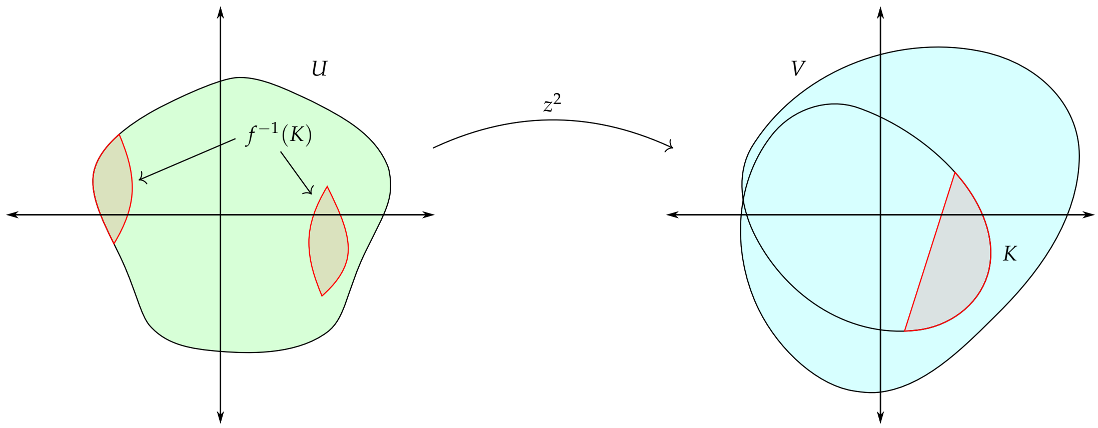


Figure 1.3: The map $f(z) = z^2$ is not proper in U.

Proof of Proposition 1.19. On the one hand, since f is a non-constant holomorphic map by the former corollary of the Open Mapping Theorem, $\partial V \subset f(\partial U)$.

On the other hand, let $x \in \partial U = \overline{U} \setminus U$ and suppose $f(x) \in \operatorname{int} V$. Then, there is an open neighbourhood W of $f(x)$ in $\operatorname{int} V$. Therefore we can choose a compact set K such that $f(x) \in K \subset W$. Now, since f is a proper map, $f^{-1}(K)$ is a compact set in U containing x, which is a contradiction because that would mean $x \in U$. Therefore, $f(x) \in \partial V$ so $f(\partial U) \subset \partial V$.

Suppose now that $f(\partial U) = \partial V$, and let $K \subset V$ be a compact set in V. Then K is closed in V, and since f is continuous, $f^{-1}(K)$ is closed in U. Moreover, since by hypothesis we know that $f(\partial U) = \partial V$, there is no point $z_0 \in \partial U$ such that $f(z_0) \in K$, so $f^{-1}(K) \subset \operatorname{int} U$. Now, since $f^{-1}(K)$ is completely inside U (hence bounded) and it is closed, it is therefore compact, which concludes the proof. $\square$

Corollary 1.21. *Any continuous function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is proper.*

Proof. Given K a compact set in $\widehat{\mathbb{C}}$, K is closed and since f is continuous, $f^{-1}(K)$ is also closed in a compact set, hence $f^{-1}(K)$ is compact and therefore f is a proper map. $\square$

Proposition 1.22. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a holomorphic map. If U is a connected component of $f^{-1}(V)$, then $f : U \to V$ is proper.*

Proof. Let $x \in \partial U = \overline{U} \setminus U$ and suppose $f(x) \in \operatorname{int} V$. Then, there is an open neighbourhood W of $f(x)$ in $\operatorname{int} V$, which means that $f^{-1}(W)$ is an open neighbourhood of x. By maximality of the connected component, the preimage $f^{-1}(W)$ must be entirely in U because W is entirely in V, but this is a contradiction because all neighbourhoods of boundary points have points in U and points in $\widehat{\mathbb{C}} \setminus U$. So $f(\partial U) \subset \partial V$, and since f is holomorphic, it is also open which means $\partial V \subset f(\partial U)$, and so the equality $f(\partial U) = \partial V$ holds. Finally by Proposition 1.19 we know that f is a proper map. $\square$

Proposition 1.23. *Let $f : U \to V$ be a non-constant holomorphic proper map, and assume V is connected. Then every $v \in V$ has the same (finite) number of preimages in U counted with multiplicity. This number is called the* degree *of $f_{|U}$, denoted by* $\deg f_{|U}$.

Proof. Since f is holomorphic, given any $v \in V$ the map $f(z) - v$ has a discrete set of zeroes and thus the fibers of f are finite. Now, let $v \in V$ and consider the fiber over it, $f^{-1}(v) = \{z_1, \dots, z_k\}$. By Theorem A.19 there exists a neighbourhood B of v and a neighbourhood W_i of z_i such that any point $\omega \in B$, $\omega \neq v$, has exactly d_i preimages in W_i, all of them regular.

Suppose that not all preimages of $\omega \in B$ are contained in $\bigcup_i W_i$. Then there exist sequences $\omega_n \to v$ and $\zeta_n \in U \setminus \bigcup_i W_i$ such that $f(\zeta_n) = \omega_n$, and since f is proper, the sequence $\{\zeta_n\}_n$ would have a limit point $\zeta^* \in U \setminus \bigcup_i W_i$. Then $f(\zeta^*) = v$ and ζ^* is different from every z_i, which is a contradiction. Therefore for B sufficiently small all preimages of $\omega \in B$ belong to $\bigcup_i W_i$, hence all points close to v have the same number of preimages counted with multiplicities as v. Then the number of preimages is locally constant, and since V is connected, this number (degree) is globally constant. □

The following equality is a generalization of the fact that the connectivity number is a conformal invariant.

Theorem 1.24. (Riemann-Hurwitz formula, [Ste, pp. 7-9]**)** *Let $f : U \to V$ be a proper map of degree k from a $m-$connected domain U onto a $n-$connected domain V, with r critical points in U counted with multiplicity. Then, the following equality, known as the* Riemann-Hurwitz formula, *is satisfied:*

$$m - 2 = k(n - 2) + r.$$

Note that for $k = 1$ and $r = 0$, we obtain $m = n$, this is, the conformal invariance of the connectivity number.

Chapter 2

Rational complex iteration

The primary aim of this chapter is to introduce some concepts of complex dynamics, specially focusing on the iteration of rational functions. This is, given a meromorphic function f and a point (seed) $z \in \mathbb{C}$, we study how the iterates $f^n(z)$ behave, for $n \geq 0$, and we classify those seeds based on its iterates under f. Results on critical points and values of f will also be analysed, as they play an important role in our later discussions. We also define the Fatou and Julia sets, and see some important and useful properties, including the Classification Theorem.

Results in this chapter can be found in [Bea, BF, CG, FJ, Mil, Ste].

2.1 Complex dynamical systems

Definition 2.1. (Dynamical system) A *dynamical system* is a triple (Ω, T, Φ), where Ω is the state space, $T \in \{\mathbb{Z}, \mathbb{R}\}$ is the time space and $\Phi : \Omega \times T \to \Omega$ the evolution law of the system. Then, for every $x \in \Omega$ and $t \in T$, the point $\Phi(x, t) \in \Omega$ gives the new state at time t given the initial state x. If $T = \mathbb{R}$ the system is called *continuous dynamical system*, whereas if $T = \mathbb{Z}$ the system is called *discrete dynamical system*, and can be seen as an iteration of a function f over a initial state x. In the second case every subsequent state can be found by applying the law function f over the initial condition a determined number of times, i.e., $f^n(x)$ would be the final state after n iterations.

Notation. From now on, we will use $f^n(z)$ to denote the function composition $f \circ \overset{n}{\cdots} \circ f\ (z)$, and $f^0 = Id$.

Some dynamical systems can seem extremely different when in reality they are closely related. The notion of conjugate dynamical systems is very useful, as it allows us to identify systems that behave essentially identically but they appear to be unalike.

Definition 2.2. (Conjugacy) Given two functions $f : X \to X$ and $g : Y \to Y$, we say they are *topologically conjugate* (often denoted by $f \underset{h}{\sim} g$) if there is a continuous and bijective map

$h : X \to Y$, called *conjugacy*, such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

Equivalently, $h(f(z)) = g(h(z))$. In this work we will be interested in conformal conjugacies (when $X, Y \subset \mathbb{C}$ and the function h is a conformal map, that is, holomorphic and bijective), but one can define other conjugacies depending on the properties of h.

Remark 2.3. The maps f and g can be seen as the same map viewed in different coordinate systems. Directly from the definition, we can apply the conjugacy to f^n and g^n and obtain that they are also h-conjugate, $g^n = h \circ f^n \circ h^{-1}$.

Definition 2.4. (Invariant set) Given a map $f : X \to X$ and a subset $U \subset X$, then U is:

(a) *forward invariant* if $f(U) \subset U$.

(b) *backward invariant* if $f^{-1}(U) \subset U$.

(c) *completely invariant* if $f(U) = U = f^{-1}(U)$.

Proposition 2.5. *Let f and g be complex maps such that $f \underset{h}{\sim} g$. Then, if U is forward (respectively backward or completely) invariant by f, $h(U)$ is forward (respectively backward or completely) invariant by g.*

Proof. Suppose that U is forward invariant by f. We want to see that for any $y \in h(U)$, $g(y) \in h(U)$. Since $y \in h(U)$, there exists $x \in U$ such that $y = h(x)$. Now, since $h \circ f = g \circ h$, we have

$$g(y) = g(h(x)) = h(f(x)) \in h(U)$$

because $f(x) \in f(U) \subset U$.

Conversely, suppose that U is backward invariant by f, and $y = h(x)$ for some $x \in U$. Then, using $g = h \circ f \circ h^{-1}$, we have

$$g^{-1}(y) = g^{-1}(h(x)) = h(f^{-1}(x)) \in h(U)$$

because $f^{-1}(x) \in f^{-1}(U) \subset U$. □

2.2 Periodic points and orbits

Definition 2.6. (Orbit) Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a complex map and $z_0 \in \widehat{\mathbb{C}}$. Then, the *forward orbit*, *positive semi-orbit* or just *orbit* is the set of forward iterates of f starting at z_0,

$$\mathcal{O}(z_0) = \mathcal{O}^+(z_0) := \{z_n := f^n(z_0)\}_{n \geq 0}.$$

The *backward orbit* or *negative semi-orbit* is the set of backwards iterates, defined as

$$\mathcal{O}^-(z_0) := \{z \mid f^n(z) = z_0 \text{ , for some } n \geq 0\} \equiv \bigcup_{n \geq 0} f^{-n}(z_0).$$

Definition 2.7. (Fixed, periodic and preperiodic points) Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a complex map. A point $z_0 \in \widehat{\mathbb{C}}$ is

(a) a *periodic point of period* $p \geq 1$ of f if $f^p(z_0) = z_0$ and $f^n(z_0) \neq z_0$ for all $n < p$.

(b) a *fixed point* of f if it is periodic with period $p = 1$.

(c) a *preperiodic point* of f if z_0 is not periodic but for some $k \geq 1$, $f^k(z_0)$ is periodic.

Notation. We will usually denote the p-periodic orbit as $\langle z_0 \rangle = \{z_0, z_1, \dots, z_{p-1}\}$.

Remark 2.8. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex map. Then all points in the periodic orbit $\langle z_0 \rangle = \{z_0, z_1, \dots, z_{p-1}\}$ satisfy $f^p(z_i) = z_i$, $0 \leq i < p$.

Example 2.9. Consider the simplest quadratic polynomial $P(z) = z^2$ in the complex sphere. Its fixed points will be those that satisfy $z^2 - z = 0$, which are $z_1 = 0$ and $z_2 = 1$. Indeed, $P(0) = 0$ and $P(1) = 1$. But are these the only fixed points of P? One could think that the former equation would give us *all* fixed points, but if we consider the point $\{\infty\}$, then it seems intuitive that since $\lim_{z\to\infty} P(z) = \infty$, ∞ should be a fixed point. To prove this, consider the conjugacy $h(z) = 1/z$. Then, we can define an equivalent dynamical system $Q = h \circ P \circ h^{-1}$, which would be $Q(z) = \frac{1}{P(1/z)} = z^2$. Then, since $h(\infty) = 0$, we see that $Q(0) = 0$ implies that $P(\infty) = \infty$, making infinity a fixed point of $P(z) = z^2$.

Proposition 2.10. *Let $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a complex polynomial of degree d. Then it has precisely $d + 1$ fixed points in $\widehat{\mathbb{C}}$, counted with multiplicity.*

Proof. Suppose $P(z) = \sum_{n=0}^{d} a_n z^n$, $a_d \neq 0$. The equation $P(z) - z = 0$, which will give us the fixed points in $\mathbb{C}$, has precisely d solutions. Then, consider the conjugacy $h(z) = 1/z$ from a neighbourhood of ∞ to a neighbourhood of 0. An equivalent dynamical system would be $Q(z) = h \circ P \circ h^{-1}(z) = \frac{1}{P(1/z)} = \frac{z^d}{\sum_{n=0}^{d} a_n z^{d-n}}$. Then, $Q(0) = 0$, which implies that $P(\infty) = \infty$, meaning that we have another fixed point. In total, we have $d + 1$ fixed points in $\widehat{\mathbb{C}}$. $\square$

Example 2.11. Following the previous example, we can also find the 2-periodic points of $P(z) = z^2$, solving $P(P(z)) = z \Rightarrow z^4 - z = 0$. Factorizing the last equation, we have $z(z-1)(z^2 + z + 1) = 0$. The first two roots are the fixed points, since a fixed point z_0 satisfies $P^n(z_0) = z_0$. So the two solutions of $z^2 + z + 1 = 0$ will be the 2-periodic points, whose values are $z_1' = e^{\frac{2\pi}{3}i}$ and $z_2' = e^{\frac{4\pi}{3}i}$. Indeed, $P(z_1') = z_2'$ and $P(z_2') = z_1'$, which gives the 2-periodic orbit: $\langle z_1' \rangle = \{z_1', z_2'\}$.

Remark 2.12. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous complex map and suppose the orbit $\{z_n\}$ satisfies $\lim\limits_{n\to\infty} z_n = z$. Then, z is a fixed point.

Proof. On the one hand, we know that $z_{n+1} \xrightarrow[n\to\infty]{} z$. On the other hand, $z_{n+1} = f(z_n)$ and when $n \to \infty$, $f(z_n) \longrightarrow \lim\limits_{n\to\infty} f(z_n) = f\left(\lim\limits_{n\to\infty} z_n\right) = f(z)$. So we finally obtain $f(z) = z$. $\square$

Definition 2.13. (Attracting and repelling fixed points) Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous complex map and z_0 be a fixed point. Then, z_0 is *stable* if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall z \in B_\delta(z_0),\ f^n(z) \in B_\varepsilon(z_0)\ \forall n \geq 0$. That is, nearby orbits cannot get far away.

A fixed point is *attracting* if it is stable and $\exists \varepsilon > 0$ such that $\forall z \in B_\varepsilon(z_0),\ f^n(z) \underset{n\to\infty}{\longrightarrow} z_0$.

A fixed point is *repelling* if it is attracting for f^{-1}, where f^{-1} is the local inverse which fixes z_0.

Definition 2.14. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous complex map and z_0 be a $p-$periodic point. Then we say it's *attracting* (resp. *repelling*) if it is attracting (resp. repelling) as a fixed point of f^p.

Remark 2.15. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous complex map and $\langle z_0 \rangle$ a p-periodic orbit of f. If some $z_i \in \langle z_0 \rangle$ are attracting (resp. repelling), then all $z_i\ \forall i \in \{0, \ldots, p-1\}$ are attracting (resp. repelling), and we say the orbit is attracting (resp. repelling).

Definition 2.16. (Multiplier) Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and z_0 be a fixed point. Then the value $\lambda = f'(z_0)$ is called the *multiplier* of f at z_0. If z_0 is a $p-$periodic point with periodic orbit $\langle z_0 \rangle = \{z_0, z_1, \ldots, z_{p-1}\}$, the *multiplier of the periodic orbit* is defined as

$$\lambda = (f^p)'(z_0) = f'(z_0) \cdot f'(z_1) \cdot \ \cdots\ \cdot f'(z_{p-1}) = \prod_{k=0}^{p-1} f'(z_k).$$

Note that for all $k \in \{0, \ldots, p-1\}$, we have $\lambda = (f^p)'(z_k)$.

Proposition 2.17. *If $f \underset{h}{\sim} g$ and z_0 is an attracting (resp. repelling) p-periodic point of f, then $h(z_0)$ is an attracting (resp. repelling) p-periodic of g. Moreover, if the conjugacy is conformal then the multiplier is preserved.*

Proof. We are going to see the case $p = 1$, this is, when z_0 is a fixed point. For $p \geq 1$ we consider z_0 as a fixed point of f^p, and the proof is analogous.

If z_0 is a fixed point of f, then $f(z_0) = z_0$. Then, since f and g are h-conjugate, we have $g(h(z)) = h(f(z))$ for every z, particularly, for z_0 we have $g(h(z_0)) = h(f(z_0)) = h(z_0)$.

Now suppose z_0 is an attracting fixed point of f. This means that $\lim_{n\to\infty} f^n(u) = z_0$ for every u in a small neighbourhood of z_0. Now we have $\lim_{n\to\infty} g^n(h(u)) = \lim_{n\to\infty} h(f^n(u)) = h\left(\lim_{n\to\infty} f^n(u)\right) = h(z_0)$, which proves that $h(z_0)$ is an attracting fixed point of g.

The repelling case is proved analogously by assuming that a repelling fixed point of f is an attracting fixed point of f^{-1}.

Finally, suppose that the conjugacy is differentiable. We previously had that $g(z) = h \circ f \circ h^{-1}(z)$, and by differentiating we obtain $g'(z) = h'(f(h^{-1}(z))) \cdot f'(h^{-1}(z)) \cdot (h^{-1})'(z)$. Now, applying the previous equality to the fixed point $h(z_0)$ of g and applying the Inverse Function Theorem, we finally obtain $g'(h(z_0)) = h'(f(z_0)) \cdot f'(z_0) \cdot (h^{-1})'(h(z_0)) = h'(z_0) \cdot f'(z_0) \cdot 1/h'(z_0) = f'(z_0)$, which proves the last result. $\square$

Theorem 2.18. (Stability of fixed points) *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, z_0 be a p-periodic point and λ its multiplier.*

(a) If $|\lambda| < 1$ then z_0 is attracting *(if $\lambda = 0$ it is called* superattracting*).*

(b) If $|\lambda| > 1$ then z_0 is repelling.

(c) If $|\lambda| = 1$ then z_0 is neutral *or* indifferent.

(c.1) If $\lambda = e^{2\pi i\alpha}$ ($\alpha = p/q \in \mathbb{Q}$) is a q^{th} root of unity then z_0 is parabolic *or* rationally indifferent.

(c.2) If $\lambda = e^{2\pi i\alpha}$ ($\alpha \in \mathbb{R} \setminus \mathbb{Q}$) then z_0 is irrationally indifferent.

Proof. We are going to see the case $p = 1$, this is, when z_0 is a fixed point. For $p \geq 1$ we consider z_0 as a fixed point of f^p, and the proof is analogous.

(a) Let ρ be such that $|\lambda| < \rho < 1$. Then, $|f(z) - z_0| \leq \rho\,|z - z_0|$ on some neighbourhood of z_0. Then iterating we get $|f^n(z) - z_0| \leq \rho^n\,|z - z_0|$, which tends to zero as n tends to infinity. So we finally get $f^n(z) \to z_0$ as $n \to \infty$.

(b) To prove that z_0 is a repelling fixed point of f we must prove that is an attracting fixed point of f^{-1}. We compute the multiplier $\tilde{\lambda}$ of f^{-1}, making use of the Inverse Function Theorem: $\tilde{\lambda} = (f^{-1})'(z_0) = \frac{1}{f'(f^{-1}(z_0))} = \frac{1}{f'(z_0)} = \frac{1}{\lambda}$, so $|\tilde{\lambda}| = \frac{1}{|\lambda|} < 1$ since $|\lambda| > 1$. Then using (a) we conclude that z_0 is an attracting fixed point of f^{-1}, thus is repelling for f. □

Example 2.19. Returning to example 2.11, we had $P(z) = z^2$, with fixed points $z_1 = 0$, $z_2 = 1$ and $z_\infty = \infty$ and 2-periodic points $z_1' = \mathrm{e}^{\frac{2\pi}{3}i}$ and $z_2' = \mathrm{e}^{\frac{4\pi}{3}i}$ which formed a periodic orbit $\langle z_1' \rangle = \{z_1', z_2'\}$. To deduce the nature of these points and orbit we compute the multiplier: $P'(z) = 2z$, which means that $P'(0) = 0$ and $P'(1) = 2$ and therefore $z_1 = 0$ is superattracting and $z_2 = 1$ is repelling. In order to study the other fixed point $z_\infty = \infty$ we have to reuse the conjugacy introduced in example 2.11, $h(z) = 1/z$. We had $Q(z) = \frac{1}{P(1/z)} = z^2$, meaning that $Q'(z) = 2z$ and hence $Q'(0) = P'(\infty) = 0$, which implies that $z_\infty = \infty$ is superattracting.

The multiplier of the 2-periodic orbit is $\lambda = 2\mathrm{e}^{\frac{2\pi}{3}i} \cdot 2\mathrm{e}^{\frac{4\pi}{3}i} = 4 > 1$ which means that is a repelling periodic orbit.

Lemma 2.20. *Let $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a complex polynomial of degree $d \geq 2$. Then ∞ is a superattracting fixed point.*

Proof. Suppose $P(z) = \sum_{n=0}^{d} a_n z^n$, $a_d \neq 0$, and consider the conjugacy $h(z) = 1/z$. An equivalent dynamical system would be $Q(z) = h \circ P \circ h^{-1}(z) = \frac{1}{P(1/z)} = \frac{z^d}{\sum_{n=0}^{d} a_n z^{d-n}}$. Then, $Q(0) = 0$, which implies that $P(\infty) = \infty$. To check its nature, differentiate $Q(z)$ to obtain

$$Q'(z) = z^{d-1} \cdot \frac{\sum_{n=0}^{d} n a_n z^{d-n}}{\left(\sum_{n=0}^{d} a_n z^{d-n}\right)^2},$$

so $Q'(0) = 0$, and therefore ∞ is a superattracting fixed point. □

Definition 2.21. (Basin of attraction) Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous complex map and z_0 be an attracting fixed point. Then we define the *basin of attraction* of z_0 as all the points z which its orbit tends to z_0, i.e.,

$$A(z_0) = \{z \in \mathbb{C} \mid f^n(z) \underset{n\to\infty}{\longrightarrow} z_0\}.$$

The connected component of $A(z_0)$ that contains z_0 is called the *immediate basin of attraction* of z_0 and is denoted by $A^*(z_0)$.

If $\langle z_0 \rangle$ is an attracting p-periodic orbit, then

$$A(\langle z_0 \rangle) = \{z \in \mathbb{C} \mid f^{np+i}(z) \underset{n\to\infty}{\longrightarrow} z_i,\ \forall i \in \{0, \dots, p-1\}\}$$

is the *basin of attraction* of the orbit $\langle z_0 \rangle$. In this case the immediate basin of attraction is defined analogously and has p connected components.

Remark 2.22. The basin of attraction of an attracting fixed point, $A(z_0)$, is an open set.

Proof. By definition of attracting fixed point, there exists an open set U around z_0 such that for any $z \in U$, $\lim\limits_{n\to\infty} f^n(z) = z_0$. Then it is easy to see that the basin of attraction is the union of the backward iterates $A(z_0) = \bigcup\limits_{n \geq 0} f^{-n}(U)$, thus $A(z_0)$ is an open set. $\square$

Example 2.23. Consider again $P(z) = z^2$. If we compute $L = \lim_{n\to\infty} P^n(z) = \lim_{n\to\infty} z^{2n}$ we see that if $|z| < 1$, then $L = 0$, but if $|z| > 1$, then $L = \infty$. This is consistent with the previous results, since the only two (super)attracting fixed points were 0 and ∞. Then we can deduce their basins of attraction:

$$A(0) = \{z \in \widehat{\mathbb{C}} \mid |z| < 1\}, \text{ and } A(\infty) = \{z \in \widehat{\mathbb{C}} \mid |z| > 1\}.$$

The boundary of $\mathbb{D}$ is completely invariant under P, since if $z = e^{i\theta}$, $|f(z)| = \left|f^{-1}(z)\right| = |z| = 1$, which means that both forward and backward iterates of $\partial\mathbb{D}$ will remain in $\partial\mathbb{D}$.

Figure 2.1 shows the dynamical planes of some quadratic polynomials $Q_c(z) = z^2 + c$, having an attracting periodic orbit.

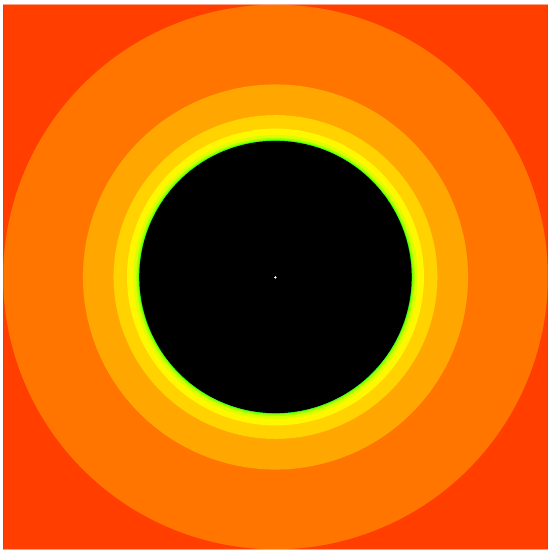
Definition 2.24. (Hyperbolic domain) We say that $U \subset \mathbb{C}$ is a *hyperbolic domain* if its complement $\mathbb{C} \setminus U$ contains at least two points.

Definition 2.25. (Absorbing domain) Let U be a hyperbolic domain in $\mathbb{C}$ and $f : U \to U$ a non-constant holomorphic map. A domain $W \subset U$ is *absorbing for f on U* if

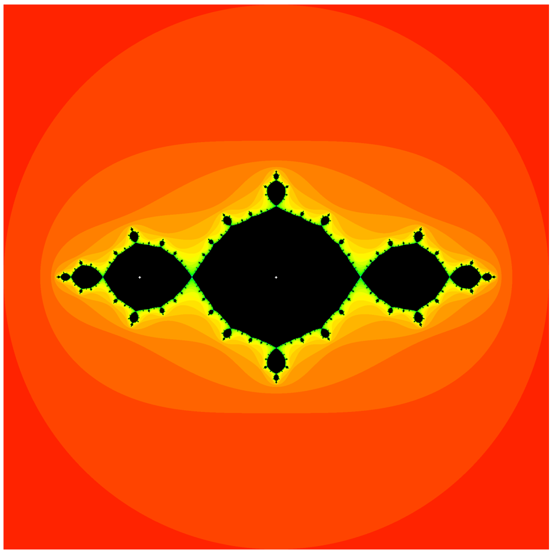
(a) $f(W) \subset W$ and

(b) for every compact set $K \subset U$ there exists a constant $n(K) \geq 0$ such that $f^n(K) \subset W$.

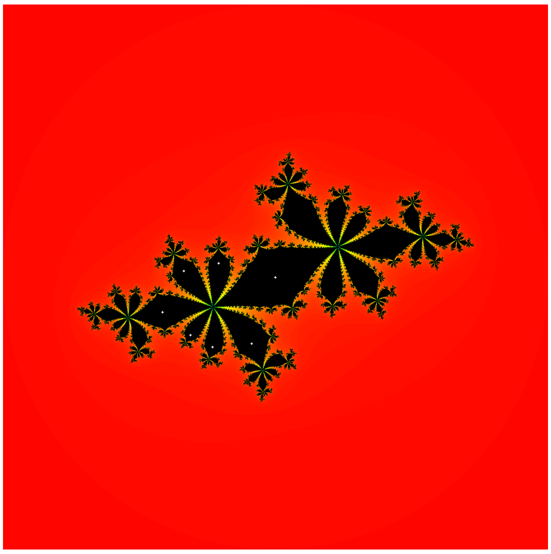
Theorem 2.26. (Kœnigs linearization Theorem) *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic complex map and z_0 be an attracting fixed point with multiplier λ satisfying $0 < |\lambda| < 1$. Then there exists a neighbourhood U of 0 and a conformal conjugacy $\zeta = \varphi(z)$ of a neighbourhood of z_0 onto U which conjugates $f(z)$ to the linear function $g(\zeta) = \lambda\zeta$. The conjugacy satisfies $\varphi(0) = 0$ and is called a* linearizing map *of f at z_0. Moreover, is unique up to multiplication by a non-zero constant.*



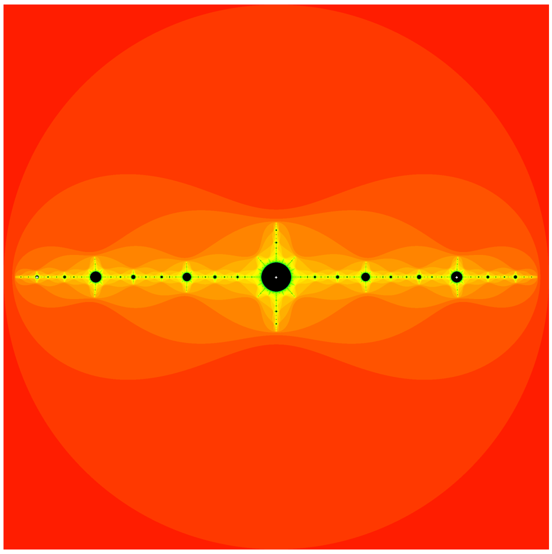
(a) $c = 0$.



(b) $c = -1$.



(c) $c = -0.6224 - 0.4249i$.



(d) $c = -1.754878$.

Figure 2.1: Different dynamical planes for the quadratic family, $Q_c(z) = z^2 + c$. The coloured region is formed by points which tend to infinity under iteration of Q_c, and the gradient of colour shows the number of iterations needed to reach a certain threshold. (a) shows the case considered in the previous examples, $Q_0(z) = z^2$. The different basins of attraction are shown in black (for the origin) and in a gradient of colours (for infinity). In (b) we see a polynomial with a periodic orbit of period 2, which is $\{0, -1\}$, whose basin of attraction is shown in black. Additionally, it has two repelling fixed points in the real axis. In (c) we see a polynomial with an attracting cycle of period 7, and in (d) a polynomial with an attracting periodic orbit of period 3. In all subfigures the attracting periodic orbit is shown in white.

Proof. Suppose that the fixed point is at the origin, $z_0 = 0$, since we can always conjugate by a translation for that to happen. In a small neighbourhood of the fixed point, $f(z) = \lambda z + O(z^2)$, and therefore $f^n(z) = \lambda^n z + O(z^2)$. Then consider $\varphi_n(z) = \frac{f^n(z)}{\lambda^n} = z + \cdots$, which satisfies

$$\varphi_n \circ f(z) = \frac{f^{n+1}(z)}{\lambda^n} = \lambda \frac{f^{n+1}(z)}{\lambda^{n+1}} = \lambda \varphi_{n+1}(z),$$

and if $\varphi_n \to \varphi$, then

$$\varphi \circ f(z) = \lambda\varphi(z) \Rightarrow \varphi \circ f \circ \varphi^{-1}(\zeta) = \lambda\zeta \text{ , and}$$
$$\varphi \circ f(0) = \varphi(0) = \lambda\varphi(0) \Rightarrow \varphi(0)(1-\lambda) = 0 \Rightarrow \varphi(0) = 0,$$

which would prove that φ is the desired conjugacy.

To prove the convergence, let $\delta > 0$ small, then

$$|f(z) - \lambda z| \leq C\,|z|^2, \quad |z| \leq \delta, \tag{2.1}$$

which leads to $|f(z)| \leq |\lambda|\,|z| + C\,|z|^2 \leq |z|\,(|\lambda| + C\delta)$ for $|z| \leq \delta$. Pick δ small enough so

$$|\lambda| + C\delta < |\lambda|^{1/2} < 1. \tag{2.2}$$

Now, suppose that $\left|f^{n-1}(z)\right| \leq |z|\,(|\lambda| + C\delta)^{n-1}$ for $|z| \leq \delta$. Then,

$$|f^n(z)| = \left|f^{n-1}(f(z))\right| \underset{(*)}{\leq} |f(z)|\,(|\lambda| + C\delta)^{n-1} \leq |z|\,(|\lambda| + C\delta)^n, \quad |z| \leq \delta, \tag{2.3}$$

where in $(*)$ we have used that $|f(z)| \leq |z|\,(|\lambda| + C\delta) \underset{(2.2)}{<} \delta$. We have proven by induction the following inequality:

$$|f^n(z)| \leq |z|\,(|\lambda| + C\delta)^n < \delta, \quad |z| \leq \delta. \tag{2.4}$$

Now, by equation (2.2), $(|\lambda| + C\delta)^2 < |\lambda|$ and therefore $\rho := \frac{(|\lambda|+C\delta)^2}{|\lambda|} < 1$. Now,

$$|\varphi_{n+1}(z) - \varphi_n(z)| = \frac{|f(f^n(z)) - \lambda f^n(z)|}{|\lambda|^{n+1}} \underset{(2.1)}{\leq} \frac{C\,|f^n(z)|^2}{|\lambda|\,|\lambda|^n} \underset{(2.4)}{\leq} \frac{C\,|z|^2\,(|\lambda| + C\delta)^{2n}}{|\lambda|\,|\lambda|^n} = \frac{C\,|z|^2\,\rho^n}{|\lambda|},$$

for $|z| \leq \delta$, and therefore φ_n converges uniformly for $|z| \leq \delta$, which proves the existence of the conjugacy.

To prove that φ is unique up to a scale factor, we have to see that any conjugation of λz is a constant multiple of z. Suppose $\varphi(z) = a_1 z + a_2 z^2 + \cdots$ and $\varphi(\lambda z) = \lambda\varphi(z)$. Substituting to the power series and comparing terms, we obtain $a_i\lambda^i = \lambda a_i \Rightarrow a_i\lambda(\lambda^{i-1} - 1) = 0$, so $a_i = 0\ \forall i \geq 2$ and thus $\varphi(z) = a_1 z$. □

Remark 2.27. The Kœnigs linearization Theorem can also be applied to repelling fixed points, since they are attracting fixed points of f^{-1}.

The special case of superattracting periodic points is due to [15]Böttcher.

Theorem 2.28. (Böttcher's Theorem, [CG, pp. 33-34]**)** *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic complex map and z_0 be a superattracting fixed point,*

$$f(z) = z_0 + a_p(z - z_0)^p + \cdots, \qquad a_p \neq 0,\ p \geq 2.$$

Then there exists a neighbourhood U of 0 and a conformal conjugacy $\zeta = \varphi(z)$ of a neighbourhood of z_0 onto U which conjugates $f(z)$ to ζ^p. The conjugacy φ is unique up to multiplication by a $(p-1)^{th}$ root of unity.

[15]Lucjan Emil Böttcher. Polish mathematician, 1872 - 1937.

Lemma 2.29. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a holomorphic map, z_0 an attracting fixed point of f, $A = A(z_0)$ its basin of attraction and U an open neighbourhood of z_0 such that $\overline{f(U)} \subset U$. Then, for every compact set $K \subset A$ there is $N \geq 0$ such that $f^N(K) \subset U$.*

Proof. By definition, since $z_0 \in \overline{f(U)} \subset U$, the basin of attraction is

$$A = A(z_0) = \bigcup_{n\geq 0} f^{-n}(U) = \bigcup_{n\geq 0} \{z \mid f^n(z) \in U\}.$$

Therefore, given any point $z \in A$ we can always find $N(z) < \infty$ such that $f^N(z) \in U$, where N is the minimum value for which this happens. We want to see than $\sup\limits_{z\in K} N(z) < \infty$.

Suppose this is not the case. Then, it must exist an infinite subsequence $\{z_k\} \subset K$ such that $N(z_k) \underset{k\to\infty}{\longrightarrow} \infty$. Every term of this infinite subsequence is in K, and since it is a compact set, its accumulation points also lay in K: $z_{k_j} \to z_* \in K \subset A$. In particular, so it does the limit term, $z_\infty \in A$, and thus there is a finite natural number N_∞ such that $f^{N_\infty}(z_\infty) \in U$. Now, using that U is an open set, we know there is a neighbourhood W of z_∞ such that $f^{N_\infty}(W) \subset U$, meaning that $N(z_k) = N_\infty$ for every z_k in W, which is a contradiction since $N(z_k) \to \infty$. Hence $\sup\limits_{z\in K} N(z) < \infty$. □

Then the following corollary is immediate.

Corollary 2.30. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a holomorphic map, z_0 an attracting fixed point of f and $A = A(z_0)$ its basin of attraction. Then, for every $z \in A$, the sequence $\{f^{-n}(z)\}_n$ is discrete in A, i.e., its accumulation points lay in ∂A.*

Lemma 2.31. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a holomorphic map and z_0 an attracting fixed point of f. Then there exists an open simply connected neighbourhood U of z_0 which is an absorbing domain for f on $A(z_0)$.*

Proof. Denoting with λ the multiplier of f, we have two cases:

(1) $\underline{0 < |\lambda| < 1}$: Then, by Kœnigs linearization Theorem f is φ-conjugate to the linear function $g(\zeta) = \lambda\zeta$ in a small neighbourhood W of 0. Now, let $\delta > 0$ such that $\mathbb{D}_\delta \subset W$, so we have:

$$g(\mathbb{D}_\delta) = \mathbb{D}_{\lambda\delta} \subset \mathbb{D}_\delta,$$

and since f and g are φ-conjugate ($\varphi \circ f = g \circ \varphi$), defining $U = \varphi^{-1}(\mathbb{D}_\delta)$, we get that U is an open simply connected neighbourhood of z_0 such that $\overline{f(U)} \subset U$.

(2) $\underline{\lambda = 0}$: Analogously, by Böttcher's Theorem f is φ-conjugate to $g(\zeta) = \zeta^p$, $p \geq 2$ in a small neighbourhood W of 0. Now, let $0 < \delta < 1$ such that $\mathbb{D}_\delta \subset W$, so we have:

$$g(\mathbb{D}_\delta) = \mathbb{D}_{\delta^p} \subset \mathbb{D}_\delta, \quad \forall p \geq 2.$$

Since f and g are φ-conjugate ($\varphi \circ f = g \circ \varphi$), defining $U = \varphi^{-1}(\mathbb{D}_\delta)$, we get that U is an open simply connected neighbourhood of z_0 such that $\overline{f(U)} \subset U$.

Finally, by using Lemma 2.29, we have that for every compact set $K \subset A(z_0)$ there is $N \geq 0$ such that $f^N(K) \subset U$, and so U is an absorbing domain for f on $A(z_0)$. □

2.3 Rational maps, critical points and values

Definition 2.32. (Rational map) A map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called a *rational map* if it is of the form

$$R(z) = \frac{P(z)}{Q(z)},$$

where both P and Q are polynomials in $\widehat{\mathbb{C}}$, not both being the zero polynomial and without common factors. The *degree* of a rational map is defined as the maximum of the degrees of P and Q, i.e., $\deg R = \max\{P, Q\}$.

Proposition 2.33. *Given a rational map R in the complex sphere with degree $d \geq 1$, every point $z_0 \in \widehat{\mathbb{C}}$ has exactly d preimages under R counted with multiplicity.*

Proof. If R is rational, $R(z) = \frac{P(z)}{Q(z)}$, and let $z_0 \in \widehat{\mathbb{C}}$. Then to compute the preimage $R^{-1}(z_0)$, we have to solve the following equation:

$$P(z) - Q(z)z_0 = 0,$$

which is a polynomial of degree d. By the Fundamental Theorem of Algebra (FTA), the equation has exactly d solutions in $\widehat{\mathbb{C}}$, counted with multiplicity. □

Proposition 2.34. *A rational map R in the complex sphere with degree $d \geq 1$ has precisely $d+1$ fixed points in $\widehat{\mathbb{C}}$ counted with multiplicity.*

Proof. If R is rational, $R(z) = \frac{P(z)}{Q(z)}$, and suppose that $d = \deg R = \deg P > \deg Q = n$. Then using the conjugacy $h(z) = 1/z$ we see that ∞ is a fixed point of R:

$$\widetilde{R}(z) = \frac{1}{R(1/z)} = \frac{Q(1/z)}{P(1/z)} = \frac{\frac{b_n + b_{n-1}z + \cdots}{z^n}}{\frac{a_d + a_{d-1}z + \cdots}{z^d}} = \frac{z^d(b_n + b_{n-1}z + \cdots)}{z^n(a_d + a_{d-1}z + \cdots)} = z^{d-n} \cdot \frac{b_n + b_{n-1}z + \cdots}{a_d + a_{d-1}z + \cdots},$$

and clearly $\widetilde{R}(0) = 0 \cdot \frac{b_n}{a_d} = 0$, meaning that $R(\infty) = \infty$. Now, the equation to find the fixed points is $P(z) - zQ(z) = 0$, which has d solutions counted with multiplicity by the FTA, and adding ∞ we obtain a total of $d+1$ fixed points.

If $d = \deg R = \deg Q \geq \deg P$, infinity is no longer a fixed point (from the discussion above), and from the FTA there are $d+1$ solutions counted with multiplicity to the equation $P(z) - zQ(z) = 0$. □

Definition 2.35. (Critical points and values) Let R be a rational map. A point $c \in \mathbb{C}$ is a *critical point* if the derivative vanishes at c. The set of critical points is denoted by

$$C(R) := \{c \in \mathbb{C} \mid R'(c) = 0\}.$$

A *critical value* is the image of a critical point, and the set of critical values is denoted by

$$V(R) := \{v \in \mathbb{C} \mid v = R(c), c \in C(R)\}.$$

Definition 2.36. (Postcritical set) The *postcritical set* of a rational map R is the set containing all critical values of every iteration of R, and it is denoted by

$$P(R) := \bigcup_{\substack{c \in C(R) \\ n \geq 0}} R^n(c).$$

Proposition 2.37. *Every rational map of degree d has at most $2d-2$ critical points in $\mathbb{C}$.*

Proof. Take $R(z) = \frac{P(z)}{Q(z)}$, then $R' = \frac{P'Q - PQ'}{Q^2} = 0$ will give us the critical points of R.

Suppose that $d = \deg R = \deg P > \deg Q = n$. Then, in order to find the critical values one has to solve $P'Q - PQ' = 0$, which is a polynomial of degree at most $d + n - 1 \leq 2d - 2$, which has at most $2d - 2$ solutions in $\mathbb{C}$ from the FTA. The case where $\deg P < \deg Q$ is analogous.

Suppose now that $d = \deg R = \deg P = \deg Q$, and let $P(z) = \sum_{i=0}^{d} a_i z^i$ and $Q(z) = \sum_{j=0}^{d} b_j z^j$ be those polynomials. Then, the equation to solve is the following:

$$0 = P(z)'Q(z) - P(z)Q'(z) = \left(\sum_{i=0}^{d} i \cdot a_i z^{i-1}\right)\left(\sum_{j=0}^{d} b_j z^j\right) - \left(\sum_{i=0}^{d} a_i z^i\right)\left(\sum_{j=0}^{d} j \cdot b_j z^{j-1}\right) =$$
$$= \sum_{i=0}^{d}\sum_{j=0}^{d} i \cdot a_i b_j z^{i+j-1} - \sum_{i=0}^{d}\sum_{j=0}^{d} j \cdot a_i b_j z^{i+j-1} = \sum_{i=0}^{d}\sum_{j=0}^{d} a_i b_j z^{i+j-1}\,(i-j) = S(z)$$

Now it's trivial to see that $\deg S \leq 2d-2$, because the highest order term, when both $i = j = d$, vanishes. By the FTA, it has at most $2d-2$ solutions. □

Proposition 2.38. *Let R be a rational map and $V(R)$ the set of critical values. If $V \subseteq \mathbb{C}$ is open and simply connected, with $v_0 \in V$, and it does not contain any critical value, i.e., $V \cap V(R) = \varnothing$, then $\forall u_0$ such that $R(u_0) = v_0$ there is a univalent (holomorphic and bijective) branch $\varphi : V \to U$ of R^{-1} such that $\varphi(v_0) = u_0$, where U is open and simply connected.*

Proof. Let U be the connected component of $R^{-1}(V) = \{z \in \mathbb{C} \mid R(z) \in V\}$ containing u_0, then U is an open set since R is continuous and V is open. By Proposition 1.22, $R : U \to V$ is a proper function, and therefore by Proposition 1.23 it has a degree, call it k, which recall is the number of preimages of R in U. Now, consider the Riemann-Hurwitz formula (1.24),

$$m - 2 = k(n-2) + r,$$

where n is one since V is simply connected. Moreover, by hypothesis there are no critical values in V and so there are no critical points in U, implying that $r = 0$. Therefore, by the aforementioned formula, we have

$$m + k = 2,$$

and since $m \geq 1$ and $k \geq 1$, the only possible solution is $m = k = 1$. This implies that U is simply connected and the degree of R is one, meaning that for every point $v \in V$, the number of preimages $f^{-1}(v)$ is one, hence $R : U \to V$ is bijective. Let $\varphi : V \to U$ be the inverse function. This map φ is holomorphic because it is the inverse function of a bijective holomorphic map, and since $R(u_0) = v_0$ we know by construction that $\varphi(v_0) = u_0$, which ends the proof. □

Corollary 2.39. *Under the same hypotheses as the previous proposition and with $U \cap P(R) = \varnothing$, then there exist exactly d^n univalent branches of R^{-n} $\forall n \geq 1$, $\varphi_n : U \to \mathbb{C}$, where $d = \deg R$.*

Proposition 2.40. *Let R be a rational map, z_0 a fixed point and $A(z_0)$ its basin of attraction. Then there are finitely many critical points in $A(z_0)$ and they all converge to z_0 under the iteration of R. Hence $P(R) \setminus \{z_0\}$ is discrete in $A(z_0)$.*

Proof. Since $\#C(R) \leq 2d - 2 < \infty$, there will be a finite number of critical points in $A(z_0)$, and since they are in the basin of attraction, its iterates will converge to z_0, the attracting fixed point. To see that $P(R) \setminus \{z_0\}$ is a discrete set in $A(z_0)$ it suffices to see that the only accumulation point of $P(R)$ in $A(z_0)$ is z_0, because given a set A, $A \setminus \operatorname{acc}(A)$ is a set of isolated points and thus discrete. But we have already discussed that, since the iterates of all critical values in $A(z_0)$ tend to z_0, a single accumulation point. Hence $P(R) \setminus \{z_0\}$ is discrete in $A(z_0)$. □

2.4 The Fatou and Julia sets

Given a holomorphic map f in the Riemann sphere, one can split the sphere into two complementary sets, called the *Julia* and *Fatou* sets. These sets can be seen as the chaotic, unstable set and the regular, stable set, because points nearby in the Julia set behave very differently when f is applied, and conversely, points nearby in the Fatou set behave similarly under f. The formal definition is as follows.

Definition 2.41. (Fatou and Julia sets) Given a holomorphic map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, the set of points $z \in \widehat{\mathbb{C}}$ such that $\{f^n\}$ is a normal family in some neighbourhood of z is called the *Fatou set*:

$$\mathcal{F}(f) = \{z \in \widehat{\mathbb{C}} \mid \{f^n\}_n \text{ is normal in some neighbourhood of } z\}.$$

Then, the *Julia set* is the complement of the Fatou set:

$$\mathcal{J}(f) = \widehat{\mathbb{C}} \setminus \mathcal{F}(f),$$

so trivially $\widehat{\mathbb{C}} = \mathcal{F}(f) \sqcup \mathcal{J}(f)$.

By definition, $\mathcal{F}(f)$ is an open set and thus $\mathcal{J}(f)$ is closed.

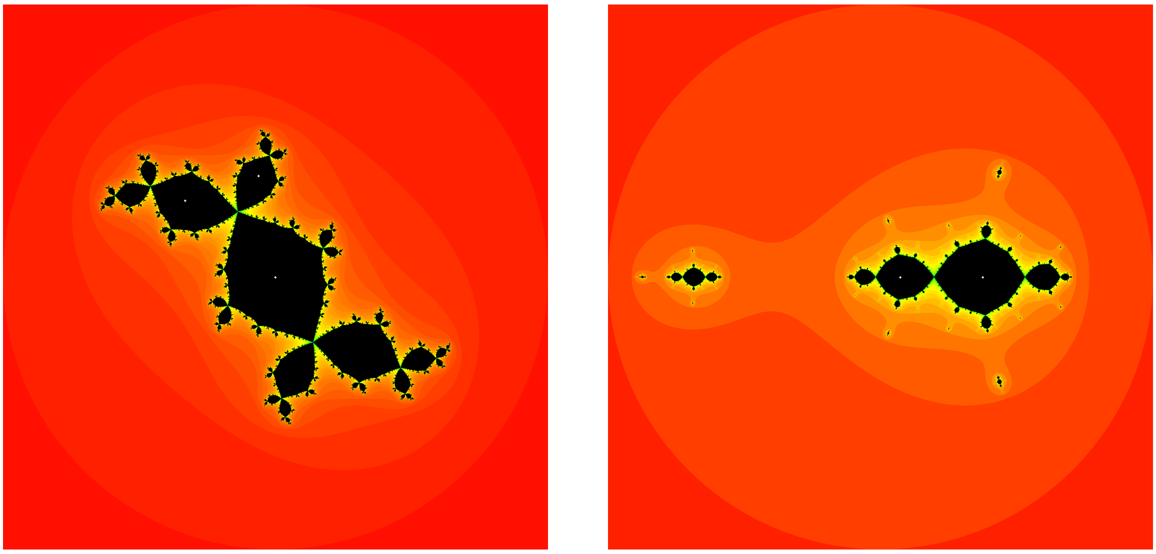
Another related set is the so called filled Julia set, defined only for polynomials.

Definition 2.42. (Filled Julia set) Given a complex polynomial P, the *filled Julia set* of P, denoted $\mathcal{K}(P)$, is the set of points that have bounded orbit, equivalently,

$$\mathcal{K}(P) := \{z \in \mathbb{C} \mid P^n(z) \nrightarrow \infty\}.$$

An equivalent definition is $\mathcal{K}(P) := \mathbb{C} \setminus A(\infty)$, where $A(\infty)$ is the basin of infinity of P.

Remark 2.43. Note that, given a polynomial P, the Julia set and the filled Julia set are directly related by $\partial\mathcal{K}(P) = \mathcal{J}(P)$.



(a) $Q_c(z) = z^2 + c$, with $c = -0.1226 + 0.7449i$.

(b) $P(z) = z^3 - 3a^2z + 1$, with $a = 0.7526$.

Figure 2.2: Dynamical planes of two polynomials. The filled Julia set is shown in black, and the boundary of this set is the Julia set of the polynomial. In (a) we see the Douady Rabbit, a quadratic polynomial with an attracting cycle of period 3. (b) shows the dynamical plane of the cubic polynomial $P(z) = z^3 - 3a^2z + 1$, with $a = 0.7526$, which has a superattracting orbit $\{a, 1 - 2a^3\}$ of period 2.

Figure 2.2 shows the dynamical plane of a quadratic and cubic polynomial with periodic orbits (in white). In black one can see the filled Julia set of each polynomial, which can be disconnected without being a Cantor set.

Theorem 2.44. (Properties of Fatou and Julia sets) *Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map with* $\deg R \geq 2$.

(1) $\mathcal{J}(R) = \mathcal{J}(R^p)$ for every $p \geq 1$, and equivalently, $\mathcal{F}(R) = \mathcal{F}(R^p)$.

(2) Both the Fatou set $\mathcal{F}(R)$ and the Julia set $\mathcal{J}(R)$ are completely invariant under R.

(3) $\mathcal{J}(R)$ contains a repelling or neutral fixed point. Hence $\mathcal{J}(R) \neq \varnothing$.

(4) Blow-up property: Given $z_0 \in \mathcal{J}(R)$, then for any neighbourhood U of z_0 the union of all iterates of U under R, $\bigcup\limits_n R^n(U)$, omits at most 2 values.

(5) $\mathcal{J}(R) = \widehat{\mathbb{C}}$ or $\operatorname{int} \mathcal{J}(R) = \varnothing$.

(6) All attracting periodic points and their basins of attraction are part of the Fatou set.

(7) All repelling periodic points are part of the Julia set, and they are dense in $\mathcal{J}(R)$.

(8) Given $z_0 \in \mathcal{J}(R)$, the set of all preimages of z_0 is dense in $z_0 \in \mathcal{J}(R)$.

(9) The boundary of every basin of attraction is in the Julia set, i.e., $\forall z_0$ attracting fixed point of R, $\partial A(z_0) \subset \mathcal{J}(R)$.

Proof.

(1) Consider the Fatou set. Since, by definition, the family of iterates $\{R^n\}_n$ is normal in all $\mathcal{F}(R)$, the subsequence $\{R^{np}\}_n \subset \{R^n\}_n$ will also be normal in $\mathcal{F}(R)$, for every $p \geq 1$, and thus $\mathcal{F}(R) \subseteq \mathcal{F}(R^p)$.

To see the reciprocal, consider now $z_0 \in \mathcal{F}(R^p)$. Then the family $\{R^{np}\}_n$ is normal in a neighbourhood U of z_0, and since R is uniformly continuous on compact sets, $\{R^{np+i}\}_n$ $\forall i \in \{0, \dots, p-1\}$ is also normal in the same neighbourhood. Observe that for any infinite subsequence of $\{R^n\}_n$ there exists a subsequence that is contained in $\{R^{np+i}\}_n$ for some value of i, i.e., $\forall \{R^{n_k}\}_k$, $\exists \{R^{n_{k_j}}\}_j$ such that $\{R^{n_{k_j}}\}_j \subset \{R^{np+i}\}_n$ for some i. Since $\{R^{np+i}\}_n$ is normal in U, it converges uniformly to some function, and thus $\{R^{n_{k_j}}\}_j$ also converges uniformly and therefore $\{R^n\}_n$ is normal in U, which proves that $z_0 \in \mathcal{F}(R)$. This concludes that $\mathcal{F}(R) = \mathcal{F}(R^p)$ and, equivalently, $\mathcal{J}(R) = \mathcal{J}(R^p)$.

(2) First, take any $\omega_0 \in \mathcal{F}(R)$, so there is an open neighbourhood U of ω_0 such that $\{R^n\}$ is normal in U. Then let z_0 be a preimage of ω_0 under R, that is, take z_0 such that $R(z_0) = \omega_0$, and let V be the connected component of $R^{-1}(U)$ containing z_0. Since R is continuous, V is an open neighbourhood of z_0. We see now that $\{R^{n+1}\}$ is normal in V because $\{R^n\}$ is normal in $R(V) = U$, and since R is uniformly continuous on compact sets, $\{R^n\}$ is also normal in V. Hence $R^{-1}(\mathcal{F}(R)) \subseteq \mathcal{F}(R)$.

Conversely, suppose $z_0 \in \mathcal{F}(R)$ and $\{R^{n_j+1}\}_j$ converges uniformly on a neighbourhood of z_0. Since R is a non-constant holomorphic map it is open, and therefore maps open neighbourhoods of z_0 onto open neighbourhoods of $R(z_0)$, meaning that $\{R^{n_j}\}_j$ converges uniformly on a neighbourhood of $R(z_0)$. Therefore $R(z_0) \in \mathcal{F}(R)$ which implies $R(\mathcal{F}(R)) \subseteq \mathcal{F}(R)$, and so $\mathcal{F}(R)$ completely invariant under R, as is $\mathcal{J}(R)$, its complement.

(4) This result is a direct consequence of Montel's Theorem. If the union of all the iterates omits more than 2 points, by Montel's Theorem $\{R^n\}$ would be a normal family in U, and $z_0 \in \mathcal{F}(R)$, which is a contradiction.

(5) Suppose that $\operatorname{int} \mathcal{J}(R) \neq \varnothing$, then there is an open set $U \subset \mathcal{J}(R)$ which is a neighbourhood of $z_0 \in \operatorname{int} \mathcal{J}(R)$. Now since the Julia set is completely invariant under R we have that $R^n(U) \subset \mathcal{J}(R)$ for all $n \geq 0$. Therefore, $\bigcup\limits_{n \geq 0} R^n(U) \subset \mathcal{J}(R)$, but by the blow-up property we know that $\bigcup\limits_{n \geq 0} R^n(U)$ omits at most 2 points. Since the Julia set is closed it follows that $\mathcal{J}(R) = \widehat{\mathbb{C}}$.

(6) We will only consider the case where z_0 is a fixed point of R, since all p-periodic points of f are fixed points of R^p. Consider z_0 an attracting fixed point and its basin of attraction $A(z_0)$, which is is an open set. Then, for each $z \in A(z_0)$ and for any neighbourhood $U \subset A(z_0)$ of z, the iterates of R in U converge to z_0. Then, $\{R^n\}$ is a normal family in U, and thus $z \in \mathcal{F}(R)$.

(7) As before, we will only consider the case where z_0 is a fixed point of R, since all p-periodic points of R are fixed points of R^p. Consider z_0 a repelling fixed point and U a neighbourhood of z_0. Then, all $z \in U$ except z_0 are moving away from U as one iterates R, at least initially. But if $z_0 \in \mathcal{F}(R)$, a subsequence of the family of iterates $\{R^{n_k}\}$ in U should converge to a holomorphic function g with $g(z_0) = z_0$. Then, since $R^{n_k} \rightrightarrows g$, we also have $(R^{n_k})' \rightrightarrows g'$ and since z_0 is repelling we have $|R'(z_0)| = \lambda > 1$. This means that $\left|(R^{n_k})'(z_0)\right| = \lambda^{n_k} \underset{n_k \to \infty}{\longrightarrow} \infty$, and thus $g'(z_0) = \infty$, which cannot happen, and therefore we reach a contradiction that proves that $z_0 \notin \mathcal{F}(R) \Rightarrow z_0 \in \widehat{\mathbb{C}} \setminus \mathcal{F}(R) = \mathcal{J}(R)$.

To see they are dense in $\mathcal{J}(R)$, for any $z_0 \in \mathcal{J}(R)$ we want to see that there is a sequence $\{z_n\}$ of periodic points such that $\{z_n\} \to z_0$. We can consider z_0 to satisfy $R(z_0) \neq z_0$ and $R'(z_0) \neq 0$, since there are finitely many of them and won't affect the following argument. Consider two different univalent branches of R^{-1}, $\varphi_1 : U \to \varphi_1(U)$ and $\varphi_2 : U \to \varphi_2(U)$, where U is an open neighbourhood of z_0 with no critical points and small enough so $\varphi_1(U) \cap \varphi_2(U) = \varnothing$. Consider now the following map,

$$g_n = \frac{R^n - \varphi_1}{\varphi_2 - \varphi_1},$$

which is well defined in U. Since U is an open neighbourhood of a point in the Julia set, $\{R^n\}$ is not normal in U, and therefore $\{g_n\}$ cannot be normal. Then, by Montel's Theorem,

$$\bigcup_n g_n(U) \nsubseteq \widehat{\mathbb{C}} \setminus \{0, 1, \infty\},$$

and so the union of these iterates must include at least one of the following points: $\{0, 1, \infty\}$.

(a) If $0 \in \bigcup_n g_n(U)$, then $g_n(z) = 0$ for some n and some z, and therefore $R^n(z) = \varphi_1(z)$ and then $R^{n+1}(z) = z$, so there is at least one periodic point in U.

(b) If $1 \in \bigcup_n g_n(U)$, then $g_n(z) = 1$ for some n and some z, and therefore $R^n(z) = \varphi_2(z)$ and then $R^{n+1}(z) = z$, so there is at least one periodic point in U.

(c) If $\infty \in \bigcup_n g_n(U)$, then $g_n(z) = \infty$ for some n and some z, and therefore $\varphi_1(z) = \varphi_2(z)$, which is a contradiction since $\varphi_1(U) \cap \varphi_2(U) = \varnothing$, so (a) or (b) must happen.

Finally, by reducing the size of U we have a sequence of periodic points $\{z_n\}$ converging to z_0. Since by Corollary 2.46 there is a finite number of attracting cycles, an infinite number of points in $\{z_n\}$ are repelling so they are in the Julia set. Hence all repelling periodic points are dense in $\mathcal{J}(R)$.

(8) We want to prove that given a point in the Julia set, the preimages of such point form a dense set in the Julia set. Suppose $z_0, \omega_0 \in \mathcal{J}(R)$, and let U be a neighbourhood of ω_0. Then, by the blow-up property, the union $\bigcup_n R^n(U)$ omits at most 2 points, meaning that there is $N \geq 1$ such that $z_0 \in R^N(U)$. Therefore there is a point $\omega^* \in U$ such that after N iterations maps to z_0, and hence it is an N^{th} preimage of z_0. If z_0 happens to be one of those 2 possibly omitted points (a finite number) the argument still holds, since there is always an infinite sequence of points $\{z_k\} \to z_0$ such that every z_k is the image of some point in U.

(9) Suppose $a \in \partial A(z_0)$. Every neighbourhood of a will have points in $A(z_0)$ and points in $\mathbb{C} \setminus A(z_0)$. Therefore we cannot find any neighbourhood such that the family of iterates $\{R^n\}$ is normal, and thus $a \in \mathcal{J}(R)$. □

Making use of some of these properties the following result can be proven.

Theorem 2.45. *Given a rational map R, every immediate basin of attraction of R contains at least one critical point.*

Proof. Suppose z_0 is an attracting fixed point. If its multiplier λ is zero, z_0 itself is a critical point and we are done. Suppose then that $0 < |\lambda| < 1$. Then, by Lemma 2.31, there is a neighbourhood U_0 of z_0 which is an absorbing domain for R, and suppose it does not contain any critical value of R. Otherwise, $R^{-1}(U_0)$ would contain a critical point and we would be done. Then, by Proposition 2.38, there is a univalent branch φ of R^{-1} satisfying $\varphi(z_0) = z_0$ and that maps U_0 into $A^*(z_0)$. Then, since U_0 is absorbing for R, we can construct a chain of inclusions $U_0 \subset \varphi(U_0) = U_1 \subset \cdots \subset \varphi(U_{n-2}) = U_{n-1} \subset \varphi(U_{n-1}) = U_n$ extending φ, where every U_k is simply connected since it is the conformal image of a simply connected set. If this procedure does not terminate we obtain a sequence $\varphi^n : U_0 \to U_n$ of analytic functions on U_0 which omits $\mathcal{J}(R)$, and so is normal on U_0. But this is a contradiction since $z_0 \in U_0$ is a repelling fixed point of φ, thus there is a U_n in which we cannot extend φ, meaning there is a critical point $z^* \in A^*(z_0)$ such that $R(z^*) \in U_n$.

If z_0 is an attracting periodic point with $p \geq 2$, the argument above holds for critical points of R^p. Since $(R^p)'(z) = \prod_{k=0}^{p-1} R'(R^k(z))$, then $A^*(z_0)$ must also contain a critical point of R. □

Corollary 2.46. *Given a rational map R of degree d, the number of attracting cycles of R is at most* $2d - 2$.

Proof. This result follows directly from the previous theorem and from Proposition 2.37. □

A very useful lemma that makes use of Bloch's Theorem (Theorem A.21) is the following.

Lemma 2.47. (Subsequences of not normal families) *Let f be a holomorphic map in a neighbourhood U of $z \in \mathcal{J}(f)$, and D an arbitrary bounded set. Then $\nexists\, n_k \to \infty$ such that $f^{n_k}(U) \subset D$, for all k.*

In order to prove the former lemma we first must state and prove a corollary of Bloch's Theorem.

Corollary 2.48. (Consequence of Bloch's Theorem) *Let f be a holomorphic function on a region containing $\overline{\mathbb{D}(z_0, r)}$. Then $f(\mathbb{D}(z_0, r)) \supset \mathbb{D}(f(z_0), R)$, where $R = \frac{1}{72} \cdot |f'(z_0)| \cdot r$.*

Proof. First, note that if $f'(z_0) = 0$ the result is trivial. Now consider the complex map defined by

$$g(z) = \frac{f(rz + z_0) - f(z_0)}{r \cdot f'(z_0)},$$

which is holomorphic in $\overline{\mathbb{D}}$. Observe that $g(0) = 0$ and $g'(z) = \frac{f'(rz+z_0)}{f'(z_0)}$, which implies that $g'(0) = 1$. We are under the assumptions of Bloch's Theorem (Theorem A.21), which ensures that $g(\mathbb{D}) \supset \mathbb{D}_L$ with $L = 1/72$. Now let $p \in \mathbb{D}(f(z_0), R)$ with $R = L \cdot |f'(z_0)| \cdot r$. This means that there is an $\omega \in \mathbb{D}_L$ such that $p = f(z_0) + f'(z_0) \cdot r \cdot \omega$. Now, since $\omega \in \mathbb{D}_L \subset g(\mathbb{D})$ it exists $z \in \mathbb{D}$ such that $g(z) = \omega$, i.e.,

$$g(z) = \frac{f(rz + z_0) - f(z_0)}{r \cdot f'(z_0)} = \omega,$$

which directly implies that

$$f(rz + z_0) = f(z_0) + r \cdot f'(z_0) \cdot \omega = p.$$

So we know so far that there exists $z \in \mathbb{D}$ such that $f(rz + z_0) = p$, hence there exists $\zeta \in \mathbb{D}(z_0, r)$ such that $f(\zeta) = p$. Since for every $p \in \mathbb{D}(f(z_0), R)$ we know there is $\zeta \in \mathbb{D}(z_0, r)$ such that $f(\zeta) = p$, we conclude that $p \in f(\mathbb{D}(z_0, r))$, which ends the proof. □

Proof of Lemma 2.47. First let $\delta > 0$ such that the disk $\mathbb{D}(z, 2\delta) \subset U$. Now, since repelling periodic points are dense in $\mathcal{J}(f)$, we know there exists a repelling periodic point of f arbitrarily close to z. Particularly, let $z_0 \in \mathbb{D}(z, \delta)$ be a repelling periodic point of f of period p, hence $z \in \mathbb{D}(z_0, \delta) \subset U$. Let $\langle z_0 \rangle = \{z_0, \ldots, z_{p-1}\}$ be the repelling orbit, whose multiplier is $\lambda = (f^p)'(z_j)$ with $|\lambda| > 1$, for any $j \in \{0, \ldots, p-1\}$.

Now we apply Corollary 2.48 to $f^{np+j}(\mathbb{D}(z_0, \delta))$. We know that $f^{np+j}(z_0) = z_j$, and that $(f^{np+j})'(z_0) = f'(z_{j-1}) \cdot \cdots \cdot f'(z_0) \cdot (f^{np})'(z_0)$ (this expression is for $j > 0$, but for $j = 0$ is trivial). Let $\mu_j = \left|f'(z_{j-1}) \cdot \cdots \cdot f'(z_0)\right| > 0$ be a finite real constant independent of n (if $j = 0$ let $\mu_0 = 1$), and note that $|(f^{np})'(z_0)| = |\lambda|^n$. Therefore, by Corollary 2.48 we get

$$f^{np+j}(\mathbb{D}(z_0, \delta)) \supset \mathbb{D}(z_j, L\delta\mu_j |\lambda|^n).$$

Since every natural number m can be written as $m = np + j$ for some n and some $j \in \{0, \ldots, p-1\}$, we know that $f^m(U) \supset f^m(\mathbb{D}(z_0, \delta))$ contains larger and larger disks as $m \to \infty$ (recall that $|\lambda| > 1$, hence $m \to \infty$ implies $n \to \infty$, and $|\lambda|^n \to \infty$). This concludes that there is no subsequence of $f^n(U)$ which remains bounded. □

2.5 Classification of Fatou components

Definition 2.49. Given a complex map f, a connected component U of the Fatou set $\mathcal{F}(f)$ is:

(a) *periodic* if for some $p \geq 1$, $f^p(U) = U$,

(b) *preperiodic* if it is not periodic but $f^p(U)$ is periodic for some $p \geq 1$,

(c) *wandering* if the sets $f^n(U)$ for $n \geq 0$ are pairwise disjoint, i.e., $\forall j, k \geq 0$ such that $j \neq k$, $f^j(U) \cap f^k(U) = \varnothing$.

Definition 2.50. (Wandering domains) Given a complex map f, a *wandering domain* of f is a wandering component of $\mathcal{F}(f)$.

The following result, conjectured by Fatou almost 100 years ago and proved by D. Sullivan in 1985 [Sul], was probably the most significant advance made in this subject in recent times and it is crucial to the proof of Theorem A.

Theorem 2.51. (No wandering domains Theorem) *Every component of the Fatou set of a rational map is eventually periodic, i.e., a rational map has no wandering domains.*

Having eliminated the possibility of wandering domains for rational functions, a natural question that could arise is how the periodic components of the Fatou set behave, or if there is a way to classify them.

Theorem 2.52. (Classification Theorem, [CG, pp. 74-79]**)** *Let R be a rational map and U a periodic component of the Fatou set $\mathcal{F}(R)$ with period $p \geq 1$. Then exactly one of the following holds:*

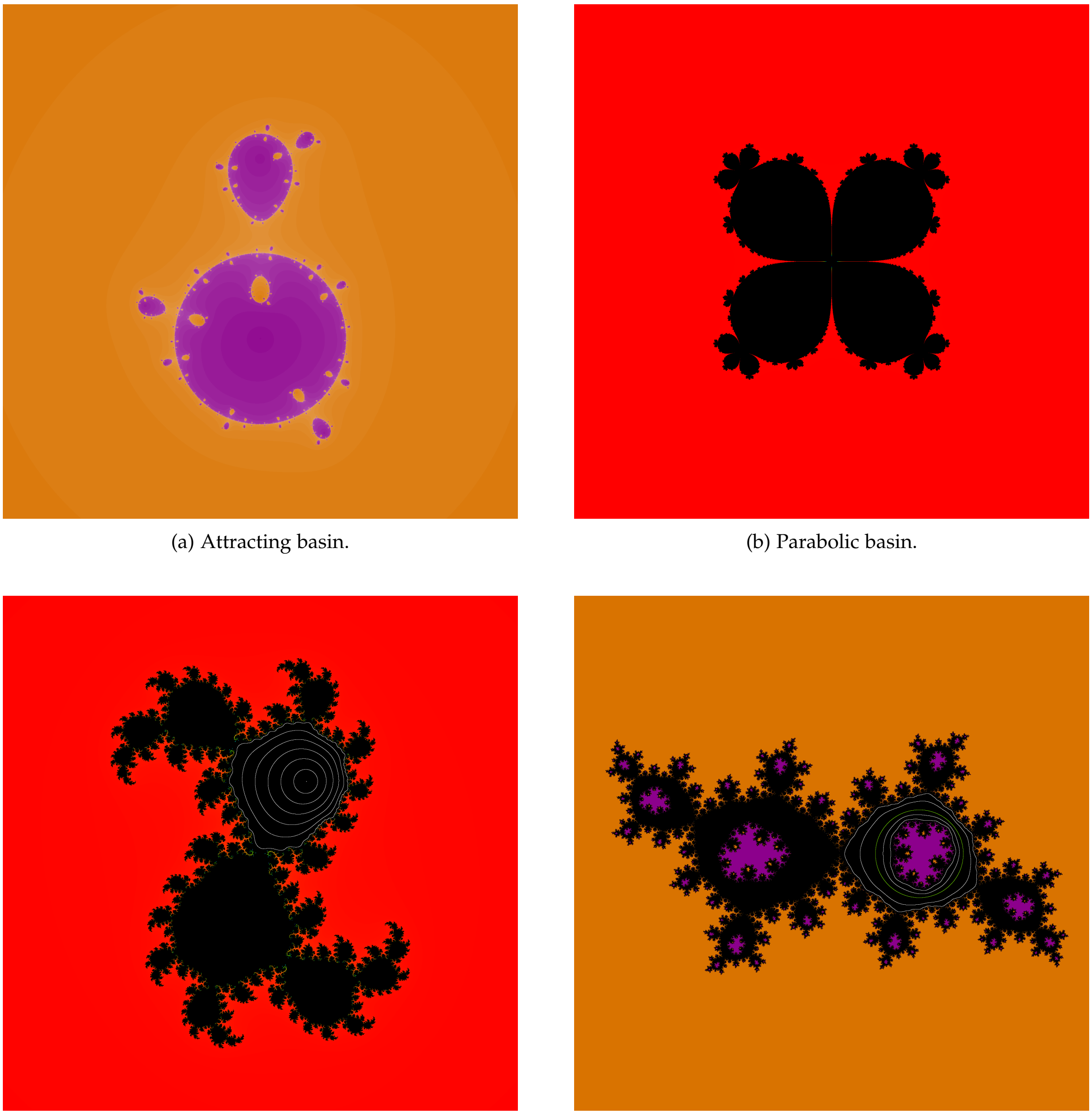
(a) U is an attracting basin, i.e., there is a point $z_0 \in U$ such that $R^{np}(z) \underset{n\to\infty}{\longrightarrow} z_0$ for all $z \in U$.

(b) U is a parabolic basin, *i.e., there is a point $z_0 \in \partial U$ such that $R^{np}(z) \underset{n\to\infty}{\longrightarrow} z_0$ for all $z \in U$.*

(c) U is a Siegel disk, *i.e., there is a point $z_0 \in U$ such that $R^p(z_0) = z_0$ and $R^p_{|U}$ is conformally conjugate to an irrational rotation, $R^p_{|U} \sim T_\theta$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$, where $T_\theta(z) = ze^{2\pi i\theta}$.*

(d) U is a Herman ring, *i.e., U is $2-$connected and $R^p_{|U}$ is conformally conjugate to an irrational rotation of the standard annulus, $R^p_{|U} \sim T_\theta$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

Examples of each Fatou component are shown in Figure 2.3. Note that, for an arbitrary rational map, the immediate basin of attraction can be bounded, and that any Fatou component can be multiply connected.



(a) Attracting basin.

(b) Parabolic basin.

(c) Siegel disk.

(d) Herman ring.

Figure 2.3: Different Fatou components according to the Classification Theorem. Each subfigure shows the dynamical plane of a different map. In (a) we see the Blaschke product $f_{\lambda,a} = z^3 \frac{z-a}{1-\bar{a}z}$ for $a = 2.1i$, where one can distinguish the basin of attraction of zero (purple) and the basin of attraction of infinity (orange). Note that, for an arbitrary rational map, the immediate basin of attraction can be bounded and multiply connected, as in the case presented. In (b) we see the map $f(z) = z + z^5$, which has four parabolic basins (black), since the origin is a parabolic fixed point of f in the Julia set. Although it is not completely visible, both axes belong to the Julia set. In (c) the quadratic map $Q_c(z) = z^2 + c$ is presented for $c \approx 0.3742 + 0.1934i$. When c is of the form $c = \frac{2e^{2\pi\theta i} - e^{4\pi\theta i}}{4}$, as is the case, $Q_c(z)$ has at least one fixed point $\omega_1 = \frac{e^{2\pi\theta i}}{2}$ which is neutral. In this particular case, $\theta = \frac{1}{2\pi}$ and $\omega_1 \approx 0.2702 + 0.4207i$. The white simple closed curves surrounding ω_1 are invariant under Q_c. (d) shows the dynamical plane for the map $f(z) = \lambda z^2 \frac{az+1}{z+a}$, where $\lambda = e^{2\pi t i}$, $t = 0.61517$ and $a = \frac{1}{4}$. It has two basins of attraction ($A(0)$ in purple and $A(\infty)$ in orange), and a Herman ring surrounding the origin (black). It is shown, in green, that the unit circle is invariant under f, and so are the white simple closed curves foliating the ring.

Chapter 3

Existence of fixed points

This chapter consists of a series of useful results to prove the existence of fixed points of rational maps. In order to develop them, we first briefly explore the concepts of polynomial-like and rational-like maps, which are the basis of complex renormalization theory. Then we state and prove some results necessary for the proof of Theorem A, many of which can be found in [BFJK1] and [BFJK2].

Most of the results of this chapter involve a certain type of fixed point, defined as follows.

Definition 3.1. (Weakly repelling fixed point) Let f be a complex map holomorphic in a neighbourhood of $z_0 \in \widehat{\mathbb{C}}$. Then z_0 is a *weakly repelling fixed point* of f if $f(z_0) = z_0$ and either $|f'(z_0)| > 1$ or $f'(z_0) = 1$.

Observe that $f'(z_0) = 1$ implies that the fixed point is double, i.e., z_0 is a double zero of $f(z) - z$. As a consequence, a small perturbation of f splits z_0 in two different fixed points, one of which is repelling.

3.1 Polynomial-like and rational-like mappings

The concept of polynomial-like mapping was first discussed by [16]Douady and Hubbard in 1985 [DH], motivated by the realisation that rational maps may locally behave like polynomials.

Definition 3.2. (Polynomial-like mapping) Let U and V be two topological disks in $\mathbb{C}$ such that $\overline{U} \subset V$, and let $f : U \to V$ be a proper holomorphic map of degree d. The triple $(f; U, V)$ is called a polynomial-like mapping of degree d.

Definition 3.3. (Filled Julia set and Julia set of a polynomial-like mapping) Let $(f; U, V)$ be a polynomial-like mapping of degree d. The filled Julia set is the set of points in U that always remain in U under iteration of f, i.e.,

$$\mathcal{K}(f) := \{z \in U \mid f^n(z) \in U,\ \forall n \geq 0\}.$$

The Julia set $\mathcal{J}(f)$ of $(f; U, V)$ is the boundary of $\mathcal{K}(f)$.

[16]Adrien Douady. French mathematician, 1935 - 2006.

Although quasiconformal mappings and quasiconformal geometry are beyond the scope of this work, we shall state and briefly discuss the Straightening Theorem to see the motivation of the former definition. Nevertheless, it will not be needed in our main proofs nor anywhere else in this thesis.

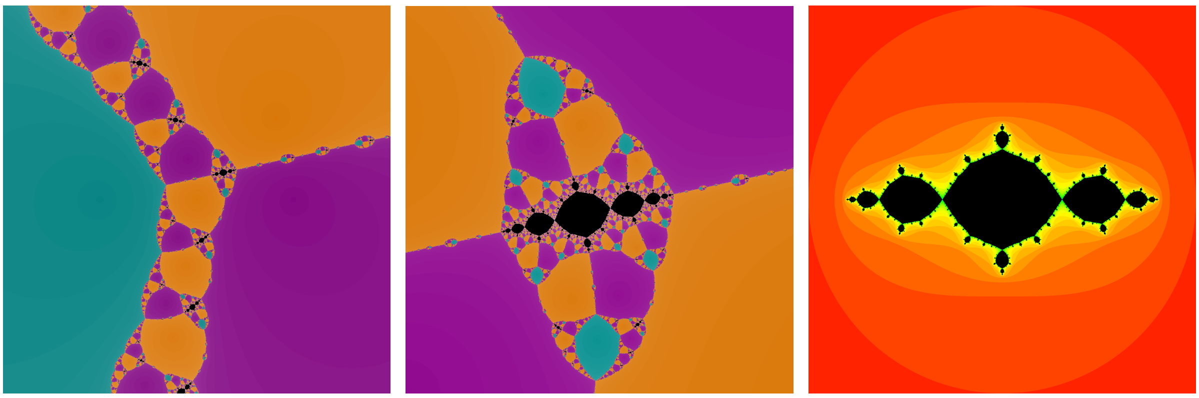
Theorem 3.4. (Straightening Theorem for polynomial-like mappings, Theorem 7.4 in [BF]**)** *Every polynomial-like mapping $(f; U, V)$ of degree d is topologically conjugate to a polynomial P of degree d in a neighbourhood of $\mathcal{K}(f)$. Moreover, if $\mathcal{K}(f)$ is connected then P is unique up to affine conjugation.*

Remark 3.5. Some observations regarding the Straightening Theorem are the following:

(1) The conjugacy is actually more regular than stated. It is in fact *quasiconformal*, which means that it distorts angles in a bounded fashion.

(2) This theorem explains why in the dynamical plane of some rational maps one can find sets that remind of the filled Julia set of a polynomial (see Figure 3.1).

(3) The property of being weakly repelling is topological, and hence is preserved under conjugacy. Therefore, since every polynomial has at least one weakly repelling fixed point, a corollary of the Straightening Theorem is that every polynomial-like mapping $(f; U, V)$ has at least one weakly repelling fixed point in U.



(a.1) (a.2) (b)

Figure 3.1: Comparison between dynamical planes of rational maps and polynomials. Subfigures (a.1) and (a.2) show the dynamical plane for the Newton map associated with the cubic polynomial $P(z) = z(z-1)(z-a)$ with $a = 0.9094 + 0.4161i$, this is, for the rational map $N(z) = z - \frac{P(z)}{P'(z)}$. Subfigure (a.2) presents an enlarged view near the point $0.65 + 0.15i$. Subfigure (b) is the dynamical plane of the quadratic map Q_{-1}, which we have already seen. Both dynamical planes are similar, and that is because there is a neighbourhood in which this Newton map is a polynomial-like map.

Analogous to the polynomial-like mappings, the definition can be extended to the case of rational-like mappings.

Definition 3.6. (Rational-like mapping) Let U and V be two domains with finite Euler characteristic in $\mathbb{C}$ such that $\overline{U} \subset V$, and let $f : U \to V$ be a proper holomorphic map of degree $d \geq 2$. The triple $(f; U, V)$ is called a rational-like mapping of degree d.

Theorem 3.7. (Straightening Theorem for rational-like mappings, Theorem 4 in [Buff]**)** *Every rational-like mapping $(f; U, V)$ of degree d is topologically conjugate to a rational map R of degree d.*

The following theorems will be used in lemmas and propositions necessary for Theorem A, as well as in the proof of the theorem itself. The proofs of these results are beyond the scope of this text, but can be found in the cited sources.

Theorem 3.8. **(**Theorem 2 in [Buff, pp. 197, 201-203]**)** *Let $D, D' \subset \mathbb{C}$ be domains with finite Euler characteristic such that $\overline{D'} \subset D$, and $f : D' \to D$ a proper holomorphic map, i.e., $(f; D', D)$ is a rational-like mapping. Then f has a weakly repelling fixed point in D'.*

This is a direct corollary of the Straightening Theorem for rational-like mappings. The following result loosens the condition $\overline{D'} \subset D$, and it is a corollary of a result proven by X. Buff, Theorem 3 in [Buff].

Theorem 3.9. **(**Corollary 2.12 in [BFJK1]**)** *Let $D \subset \widehat{\mathbb{C}}$ be a simply connected domain with locally connected boundary and $D' \subset D$ a domain with finite Euler characteristic. Let f be a continuous map on $\overline{D'} \subset \widehat{\mathbb{C}}$, meromorphic in D' such that $f : D' \to D$ is proper. Then,*

(a) if $\deg f > 1$ and f has no fixed points in $\partial D \cap \partial D'$, or

(b) if $\deg f = 1$ and $D \neq D'$,

then f has a weakly repelling fixed point in D'.

3.2 Tools to find fixed points of rational maps

This section contains some useful results for proving Theorem A. Most of the results can be found in [BFJK2]. For that purpose we must start with some definitions and notations used throughout this section.

Notation. Let $X \subset \mathbb{C}$ be a compact set. We denote by $\operatorname{ext}(X)$ the connected component of $\widehat{\mathbb{C}} \setminus X$ containing infinity. Let $K(X)$ be the closed and bounded set given by

$$K(X) := \widehat{\mathbb{C}} \setminus \operatorname{ext}(X).$$

For a Jordan curve $\gamma \subset \mathbb{C}$ we denote by $\operatorname{int}(\gamma)$ the bounded component of $\mathbb{C} \setminus \gamma$.

Lemma 3.10. *Let $K \subset \mathbb{C}$ be a compact set and U be a connected component of $\widehat{\mathbb{C}} \setminus \partial K$. Then either*

$$U \subset K \quad \text{or} \quad U \cap K = \varnothing.$$

Proof. First observe that since ∂K is compact, every connected component U of $\widehat{\mathbb{C}} \setminus \partial K$ is open. Suppose U contains points in K and in $\widehat{\mathbb{C}} \setminus K$, so let $z_1, z_2 \in U$ such that $z_1 \in K$ and $z_2 \notin K$. Then since U is open and connected there is a continuous curve $\gamma \subset U$ joining z_1 and z_2. This implies there is a point $\omega \in \gamma \subset U$ such that $\omega \in \partial K$, which is a contradiction. □

Corollary 3.11. *Let $X \subset \mathbb{C}$ be a compact set, and f a meromorphic function in a neighbourhood of $K(X)$. Let U be a connected component of $\widehat{\mathbb{C}} \setminus \partial f(X)$. Then either*

$$U \subset f(X) \quad \text{or} \quad U \cap f(X) = \varnothing.$$

Proof. Apply the previous lemma with the compact set $K = f(X)$. □

Lemma 3.12. *Let $X \subset \mathbb{C}$ be a compact set. Then,*

(a) $K(X)$ *is compact and* $\widehat{\mathbb{C}} \setminus K(X) = \operatorname{ext}(X)$ *is connected.*

(b) if $Y \subset X$ *is compact, then* $\operatorname{ext}(X) \subset \operatorname{ext}(Y)$ *and* $K(Y) \subset K(X)$.

(c) every connected component of $K(X)$ *is full.*

(d) $X \subseteq K(X)$ *and* $X = K(X)$ *if and only if every connected component of* X *is simply connected.*

(e) if X *is connected,* $\operatorname{ext}(X) \subset \widehat{\mathbb{C}}$ *is simply connected and* $K(X) \subset \mathbb{C}$ *is also connected.*

(f) if X *has a finite number of components, then* $\operatorname{ext}(X)$ *has finite Euler characteristic.*

(g) $\operatorname{ext}(K(X)) = \operatorname{ext}(X)$ *and* $K(K(X)) = K(X)$.

(h) if f *is a meromorphic function in a neighbourhood of* $K(X)$ *and* $K(X)$ *does not contain poles of* f*, then* $f(K(X)) \subset K(f(X))$.

Proof.

(a) If X is compact in $\mathbb{C}$ it is closed and bounded, therefore $\operatorname{ext}(X)$ is open and unbounded and $K(X) = \widehat{\mathbb{C}} \setminus \operatorname{ext}(X)$ is closed and bounded, hence compact. Moreover $\widehat{\mathbb{C}} \setminus K(X) = \operatorname{ext}(X)$ is connected by definition.

(b) If $Y \subset X$ then $\widehat{\mathbb{C}} \setminus X \subset \widehat{\mathbb{C}} \setminus Y$, particularly for the connected component containing infinity, hence $\operatorname{ext}(X) \subset \operatorname{ext}(Y)$. Consequently $\widehat{\mathbb{C}} \setminus \operatorname{ext}(Y) \subset \widehat{\mathbb{C}} \setminus \operatorname{ext}(X)$, that is, $K(Y) \subset K(X)$.

(c) This follows directly from the fact that $\operatorname{ext}(X)$ is connected.

(d) Let $z \in X$, so $z \notin \operatorname{ext}(X)$ and then $z \in K(X) = \widehat{\mathbb{C}} \setminus \operatorname{ext}(X)$. Now, if every connected component of X is simply connected, $\widehat{\mathbb{C}} \setminus X$ is connected and since it is the only component and contains infinity, it is exactly $\operatorname{ext}(X)$. Hence $X = \widehat{\mathbb{C}} \setminus \operatorname{ext}(X) = K(X)$.

(e) Let $X \subset \mathbb{C}$ be a connected compact set, and suppose $\operatorname{ext}(X)$ is $n-$connected. Then the union of connected components T_j of $\widehat{\mathbb{C}} \setminus \operatorname{ext}(X)$ contains X, hence it is possible to find sets $A_j \subset T_j$ such that $X = \bigcup_j A_j$. Now, since the components T_j are disconnected from one another we know that $\overline{A_j} \cap \overline{A_k} = \varnothing$ for all $j \neq k$, which is a contradiction since X is connected. Now, since $\operatorname{ext}(X)$ is simply connected, its complement $K(X)$ is also simply connected since they share one single connected boundary.

(f) If X has $k < \infty$ connected components, then $\operatorname{ext}(X)$ is $k-$connected, and thus $\chi(\operatorname{ext}(X)) = 2 - k$ is finite (see Proposition A.24).

(g) By definition $\operatorname{ext}(X) = \widehat{\mathbb{C}} \setminus K(X)$ is connected. Now, since $K(X)$ is bounded $\widehat{\mathbb{C}} \setminus K(X)$ contains infinity and therefore $\operatorname{ext}(K(X)) = \operatorname{ext}(X)$. Finally, $K(K(X)) = \widehat{\mathbb{C}} \setminus \operatorname{ext}(K(X)) = \widehat{\mathbb{C}} \setminus \operatorname{ext}(X) = K(X)$.

(h) First notice that X, $f(X)$, $K(f(X))$ and $f(K(X))$ are compact sets. To see the inclusion $f(K(X)) \subseteq K(f(X))$ we have to see that for every $z \in K(X)$, its image $f(z)$ is in $K(f(X)) = \widehat{\mathbb{C}} \setminus \operatorname{ext}(f(X))$, i.e., $\forall z \in K(X)$, $f(z) \notin \operatorname{ext}(f(X))$. By definition this holds for every $z \in X$, since $f(z) \in f(X)$. Suppose now that $z \in U$, where U is a connected component of $K(X) \setminus X$, so $f(z)$ is in a connected component of $\widehat{\mathbb{C}} \setminus \partial f(X)$. If $f(z)$ is in the connected component of $\widehat{\mathbb{C}} \setminus \partial f(X)$ containing infinity, namely $\operatorname{ext}(f(X))$, this means the connected component U of $\widehat{\mathbb{C}} \setminus \partial X$ containing z is mapped onto $\operatorname{ext} f(X)$, so there would be a pole in $U \subset K(X)$, which is a contradiction.

□

Remark 3.13. Note that given $X \subset \mathbb{C}$ and f under assumptions of Lemma 3.12 (h), the other inclusion $f(K(X)) \supset K(f(X))$ is not always fulfilled. For instance, take the analytic map $f(z) = z^2$ and the compact set

$$X = \{x \in \mathbb{C} \mid r \leq |z| \leq 1, -\pi + \theta \leq \operatorname{Arg}(z) \leq \pi - \theta\},$$

where $0 < r < 1$ and $0 < \theta < \frac{\pi}{2}$. Now since X is simply connected we have $K(X) = X$. Moreover $f(X) = \overline{\mathbb{A}_{r^2}}$, hence $K(f(X)) = \overline{\mathbb{D}}$. Then it is clear that $0 \in K(f(X))$, but $0 \notin f(K(X))$ (see Figure 3.2).

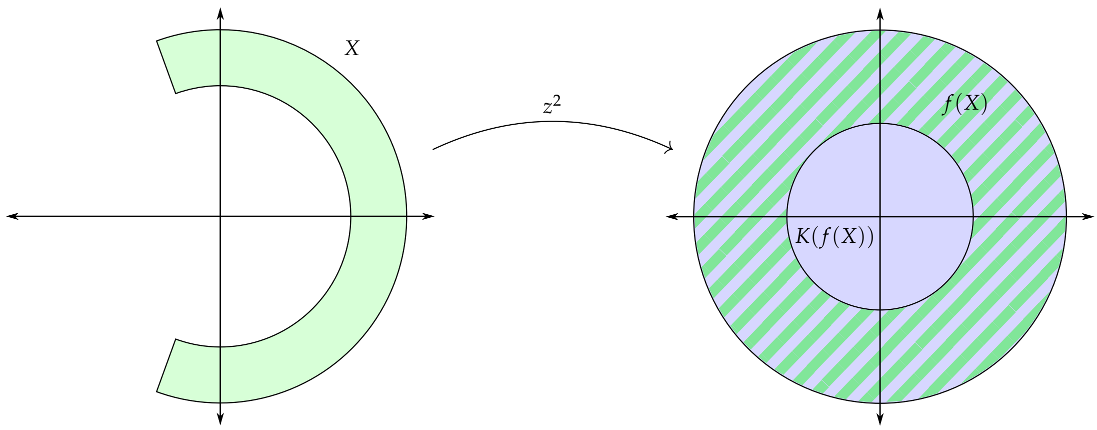


Figure 3.2: The inclusion $f(K(X)) \supset K(f(X))$ is not always satisfied. In the figure above, X is the green set, $f(X)$ is the green-striped set and $K(f(X)$ is the blue set.

Lemma 3.14. (Poles in loops) *Let $R : \mathbb{C} \to \widehat{\mathbb{C}}$ be a rational map for which infinity belongs to the Julia set. Let $\gamma \subset \mathbb{C}$ be a closed curve in a Fatou component U of R such that $K(\gamma) \cap \mathcal{J}(R) \neq \varnothing$. Then there exists an $n \geq 0$ such that $K(R^n(\gamma))$ contains a pole of R. Consequently, if U is multiply connected, there exists a bounded component of $\widehat{\mathbb{C}} \setminus R^n(U)$ which contains a pole.*

Proof. Let $\gamma \in \mathbb{C}$ be a closed curve in a Fatou component U of R such that $K(\gamma) \cap \mathcal{J}(R) \neq \varnothing$. Now, we know from 2.44 that preimages of a given point in the Julia set are dense in the Julia set. Since $\infty \in \mathcal{J}(R)$, we know there are prepoles accumulating at every point of $\mathcal{J}(R)$. Let $n \geq 0$ be the smallest order among all the prepoles in $K(\gamma)$. Since n is minimal we know that for all $1 \leq j \leq n$ the function R^j has no poles in $K(\gamma)$, and thus since R^j is meromorphic in $\mathbb{C}$, by Lemma 3.12 we have $R^j(K(\gamma)) \subset K(R^j(\gamma))$, meaning that $K(R^n(\gamma))$ contains a pole of R. Additionally, since $R^n(\gamma) \subset R^n(U)$, it follows that the pole belongs to a bounded component of $\widehat{\mathbb{C}} \setminus R^n(U)$. □

The following is a useful topology result used widely in the succeeding results as well as in the proof of Theorem A. Its proof is outside the scope of this thesis, but can be found in the cited source.

Theorem 3.15. (Torhorst Theorem, Theorem 2.2 in [Why, pp. 106-107]**)** *Let $X \subset \widehat{\mathbb{C}}$ be a locally connected continuum. Then the boundary of every component of $\widehat{\mathbb{C}} \setminus X$ is itself a locally connected continuum.*

The following results are build upon Theorems 3.8 and 3.9.

Lemma 3.16. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with finite Euler characteristic and let f be a meromorphic map in a neighbourhood of $\overline{\Omega}$. Assume there exists a component D of $\widehat{\mathbb{C}} \setminus f(\partial\Omega)$ such that*

(a) there exists $z_0 \in \Omega$ such that $f(z_0) \in D$, and

(b) $\overline{\Omega} \subset D$.

Then f has a weakly repelling fixed point in Ω.

The assumption (b) in the lemma above can be replaced by other assumptions if, additionally, Ω is simply connected with locally connected boundary.

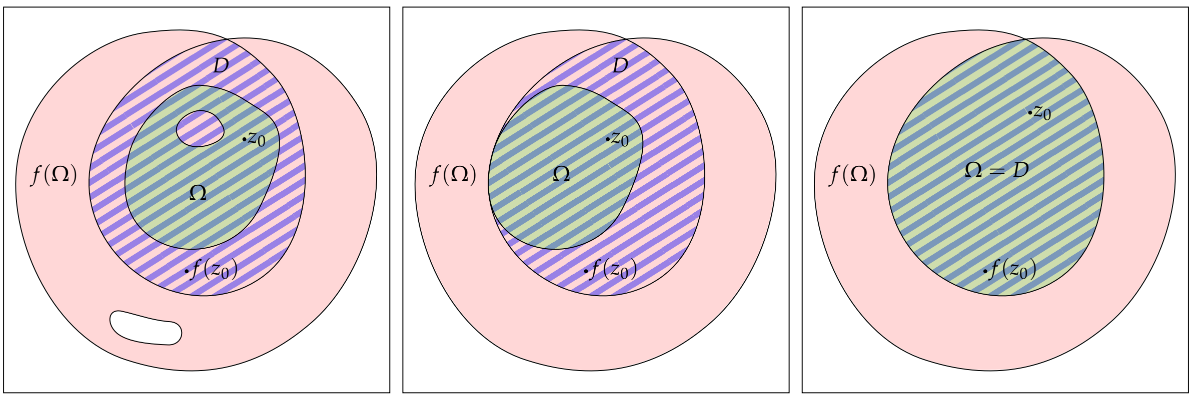
Lemma 3.17. *Let $\Omega \subset \mathbb{C}$ be a simply connected bounded domain with locally connected boundary and with finite Euler characteristic, and let f be a meromorphic map in a neighbourhood of $\overline{\Omega}$. Assume there exists a component D of $\widehat{\mathbb{C}} \setminus f(\partial\Omega)$ such that*

(a) there exists $z_0 \in \Omega$ such that $f(z_0) \in D$, and

(b.1) $\Omega \subsetneq D$ and f has no fixed points in $\partial\Omega \cap f(\partial\Omega)$, or

(b.2) $\Omega = D$, f has no fixed points in $\partial\Omega$ and $f(\Omega) \neq \Omega$.

Then f has a weakly repelling fixed point in Ω.



(a) (b) (c)

Figure 3.3: Possible setups of Ω (green), $f(\Omega)$ (red) and D (blue stripes) in lemmas 3.16 and 3.17. Subfigure (a) corresponds to the assumptions of Lemma 3.16 and subfigures (b) and (c) to the assumptions (b.1) and (b.2), respectively, of Lemma 3.17.

Proof of Lemma 3.16. By assumption (a) there exists a component D' of $f^{-1}(D)$ containing z_0. Then, since $z_0 \in \Omega \cap D'$, either $D' \subset \Omega$ or $D' \cap \partial\Omega \neq \varnothing$. Suppose there exists $z \in D' \cap \partial\Omega$, then

$$f(z) \in f(D' \cap \partial\Omega) \subset f(D') \cap f(\partial\Omega) = D \cap f(\partial\Omega),$$

which is a contradiction since D is a component of $\widehat{\mathbb{C}} \setminus f(\partial\Omega)$ and so $D \cap f(\partial\Omega) = \varnothing$. Hence $D' \subset \Omega$, which ensures that D' is bounded.

Moreover, as Ω has finite Euler characteristic, it is $k-$connected (for some finite value of k), and so $\partial\Omega$ has a finite number of connected components, as well as $f(\partial\Omega)$ and ∂D. Therefore D has finite Euler characteristic. Furthermore, since D' is a connected component of $f^{-1}(D)$, the restriction $f : D' \to D$ is proper, and hence $f_{|D'}$ has a finite degree, which implies that D' has also finite Euler characteristic.

Now, from assumption (b) we know that $\overline{D'} \subset \overline{\Omega} \subset D$, and thus $f : D' \to D$ satisfies the hypotheses of Theorem 3.8. Hence f has a weakly repelling fixed point in $D' \subset \Omega$. $\square$

Proof of Lemma 3.17. From the argument given in the former proof we already know that there is a connected component D' of $f^{-1}(D)$ containing z_0 such that $D' \subset \Omega$, that both D' and D have finite Euler characteristic and that $f : D' \to D$ is a proper meromorphic map in a neighbourhood of $\overline{\Omega}$ (hence meromorphic and continuous in $\overline{D'}$).

Moreover, we also know that, since Ω is simply connected with locally connected boundary, $\partial\Omega$ (and hence $f(\partial\Omega)$) is a locally connected continuum in $\widehat{\mathbb{C}}$, and from that follows that D is simply connected. By the Torhorst Theorem, since $f(\partial\Omega)$ is a locally connected continuum and D is a connected component of $\widehat{\mathbb{C}} \setminus f(\partial\Omega)$, the boundary of D is locally connected.

Assume now the conditions given by (b.1) are met. Then, since $D' \subset \Omega \subsetneq D$ and $\partial D \subset f(\partial\Omega)$, thus either

$$\partial D' \cap \partial D = \varnothing \quad \text{or} \quad \partial D' \cap \partial D \subset \partial\Omega \cap f(\partial\Omega)$$

cap happen. In both scenarios there are no fixed points in $\partial D' \cap \partial D$, since by the conditions given by (b.1) f has no fixed points in $\partial\Omega \cap f(\partial\Omega)$. Furthermore, $D' \subsetneq D$ implies that $D' \neq D$, hence all the assumptions in Theorem 3.9 are satisfied, providing the existence of a weakly repelling fixed point of f in $D' \subset \Omega$.

On the other hand, suppose (b.2) is satisfied, so it follows that $D' \subsetneq D$. Otherwise, if $D' = D = \Omega$, we would have $f(\Omega) = \Omega$, which is a contradiction stated in (b.2). Additionally, f has no fixed points in $\partial\Omega = \partial D$, meaning it has no fixed points in $\partial D' \cap \partial D$. Finally, since $D' \neq D$ we are again under the assumptions of Theorem 3.9, hence f has a weakly repelling fixed point in $D' \subset \Omega$, which ends the proof. $\square$

These two results provide the following corollaries.

Corollary 3.18. *Let $X \subset \mathbb{C}$ be a continuum and f a meromorphic map in a neighbourhood of $K(X)$. Suppose that*

(a) f has no poles in X,

(b) $K(X)$ contains a pole of f,

(c) $K(X) \subset \operatorname{ext}(f(X))$.

Then f has a weakly repelling fixed point in the interior of $K(X)$.

Proof. First of all, assumption (a) ensures that, since X is a continuum (non-empty compact connected set of $\mathbb{C}$), $f(X)$ and hence $K(f(X))$ are continua in $\mathbb{C}$. Now let $p \in K(X)$ be a pole of f. Again, assumption (a) ensures that there is a bounded simply connected open component Ω of $\widehat{\mathbb{C}} \setminus X$ such that

$$p \in \Omega \subset \overline{\Omega} \subset K(X). \tag{3.1}$$

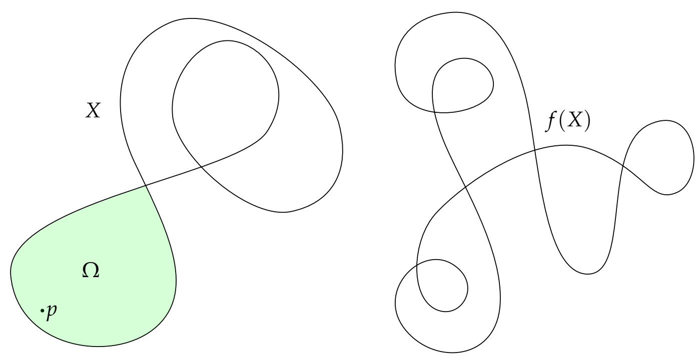


Figure 3.4: Setup of X, $f(X)$, Ω and p in the proof of Corollary 3.18.

Therefore $\partial\Omega \subset X$, hence $f(\partial\Omega) \subset f(X)$, and so by assumption (c) and Lemma 3.12 we have

$$K(X) \subset \operatorname{ext}(f(X)) \subset \operatorname{ext}(f(\partial\Omega)).$$

This fact together with the chain of inclusions in (3.1) implies that $\overline{\Omega} \subset K(X) \subset \operatorname{ext}(f(\partial\Omega))$.

Finally, let $D = \operatorname{ext}(f(\partial\Omega))$, which by definition is a component of $\widehat{\mathbb{C}} \setminus f(\partial\Omega)$ containing infinity, such that $\overline{\Omega} \subset D$, $p \in \Omega$ and $f(p) = \infty \in D$. Since f is meromorphic in a neighbourhood of $\overline{\Omega} \subset K(X)$, we are under the assumptions of Lemma 3.16, which provides a weakly repelling fixed point of f in $\Omega \subset K(X)$, hence in the interior of $K(X)$. □

Corollary 3.19. *Let $X \subset \mathbb{C}$ be a continuum and f a meromorphic map in a neighbourhood of $X \cup K(f(X))$. Suppose that*

(a) f has no poles in X,

(b) $X \subset K(f(X))$,

(c) $f^2(X) \subset \operatorname{ext}(f(X))$.

Then f has a weakly repelling fixed point in the interior of $K(f(X))$.

Proof. Similarly to the former proof, assumption (a) ensures that, since X is a continuum, $f(X)$ and hence $K(f(X))$ are continua in $\mathbb{C}$, and $f^2(X)$ is a continuum in $\widehat{\mathbb{C}}$ because $f(X)$ could contain a pole of f. Furthermore, $X \cap f(X) = \varnothing$. Notice that, otherwise, if $z \in X \cap f(X)$ then

$$f(z) \in f(X \cap f(X)) \subset f(X) \cap f^2(X),$$

which contradicts assumption (c), that states $f(X) \cap f^2(X) = \varnothing$. Then, (b) provides a bounded simply connected open component Ω of $\widehat{\mathbb{C}} \setminus f(X)$ such that

$$X \subset \Omega \subset \overline{\Omega} \subset K(f(X)). \tag{3.2}$$

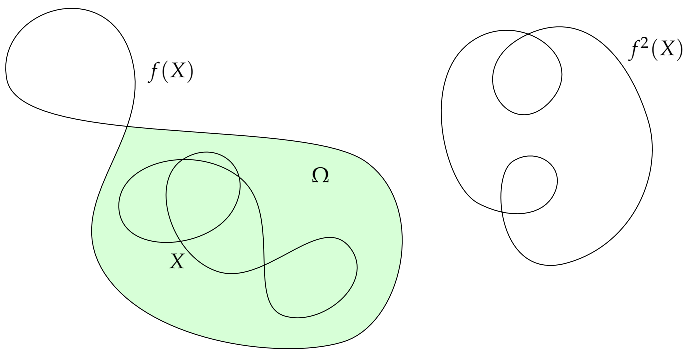


Figure 3.5: Setup of X, $f(X)$, $f^2(X)$ and Ω in the proof of Corollary 3.19.

Therefore $\partial\Omega \subset f(X)$ and $f(\partial\Omega) \subset f^2(X)$, so by assumption (c) we have

$$\widehat{\mathbb{C}} \setminus \mathrm{ext}(f(X)) = K(f(X)) \subset \widehat{\mathbb{C}} \setminus f^2(X) \subset \widehat{\mathbb{C}} \setminus f(\partial\Omega).$$

It follows that there is some component D of $\widehat{\mathbb{C}} \setminus f(\partial\Omega)$ such that $K(f(X)) \subset D$. This fact together with the chain of inclusions in (3.2) implies that $\overline{\Omega} \subset K(f(X)) \subset D$. Moreover for any $z_0 \in X \subset \Omega$ we have $f(z_0) \in f(X) \subset D$. Since f is meromorphic in a neighbourhood of $\overline{\Omega} \subset K(f(X))$, we are again under the assumptions of Lemma 3.16, which provides a weakly repelling fixed point of f in $\Omega \subset K(f(X))$, hence in the interior of $K(f(X))$. □

The following theorem will be very useful in order to find fixed points of a meromorphic function in a bounded region, assuming we know how many poles it has in that region. It is important to keep in mind that the multiplicity of a fixed point z_0 of f is the multiplicity of z_0 as a zero of the function $f(z) - z$.

Theorem 3.20. *Let $\Omega \subset \mathbb{C}$ be an open simply connected bounded domain and let f be a meromorphic map in a neighbourhood of $\overline{\Omega}$ such that $f(\partial\Omega) \subset \Omega$. Let m be the number of poles of f in Ω counted with multiplicities. Then Ω contains exactly $m+1$ fixed points of f, counted with multiplicities.*

Theorem 3.20, unlike previous lemmas and corollaries, does not state the nature of the fixed points. Nevertheless, it will be of great importance in the proof of Theorem A. In order to prove it, we first have to develop some results about winding numbers, whose definition and basic properties can be found in Appendix A.1.

Lemma 3.21. *Let $\gamma, \sigma \subset \mathbb{C}$ be two disjoint closed curves and let $P \in \gamma$, $Q \in \sigma$ be arbitrary points on these curves (see Figure 3.6). Then, the following equality holds:*

$$\mathrm{wind}(\sigma(t) - \gamma(t), 0) = \mathrm{wind}(\gamma, Q) + \mathrm{wind}(\sigma, P).$$

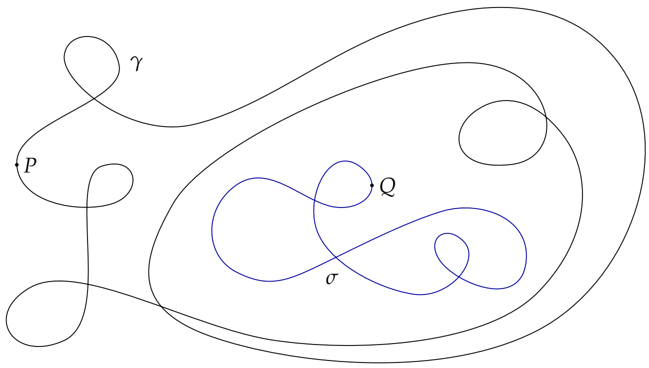


Figure 3.6: Layout of $P \in \gamma$ and $Q \in \sigma$ in Lemma 3.21.

Proof. First suppose that γ belongs to a bounded component of $\mathbb{C} \setminus \sigma$. Then, the left side of the equality is invariant under homotopies of γ in $\mathbb{C} \setminus \sigma$. This means we can contract γ to the constant curve $P \in \gamma$, obtaining

$$\mathrm{wind}(\sigma(t) - \gamma(t), 0) = \mathrm{wind}(\sigma(t) - P, 0) = \mathrm{wind}(\sigma(t), P),$$

and since $\mathrm{wind}(P, Q) = 0$, the equality is proved. We can proceed analogously for the symmetric case, when σ belongs to a bounded component of $\mathbb{C} \setminus \gamma$. The remaining case, when both curves are not bounded by the other, is trivial since both are contractible. $\square$

The following is simply a corollary of the Argument Principle.

Corollary 3.22. (Corollary of the Argument Principle) *Let $\Omega \subset \mathbb{C}$ be an open domain bounded by a Jordan curve $\gamma \subset \mathbb{C}$ and let f be a meromorphic map in a neighbourhood of $\overline{\Omega}$ such that $f(z) \notin \{z, \infty\}$ for all points in $z \in \gamma$. Let $\mathrm{Fix}\,(f)$ and $P(f)$ be the sets of fixed points and poles of f, respectively. Then,*

$$\mathrm{wind}(f(\gamma(t)) - \gamma(t), 0) = \#(\mathrm{Fix}\,(f) \cap \Omega) - \#(P(f) \cap \Omega),$$

where fixed points and poles are counted with multiplicities, and # denotes cardinality.

Proof. This result follows directly from the Argument Principle applied to the map $f - \mathrm{Id}$ and, since the domain is bounded by a Jordan curve, $\mathrm{wind}(\gamma, z) = 1$ for every $z \in \Omega$, particularly for poles and fixed points. $\square$

We now have the necessary tools to prove Theorem 3.20.

Proof of Theorem 3.20. Since $f(\partial\Omega) \subset \Omega$ and Ω is open and bounded we know $\partial\Omega$ does not contain neither fixed points nor poles of f. Therefore, since fixed points and poles are isolated in $\mathbb{C}$, it follows there are no poles in a small neighbourhood of $\partial\Omega$,

$$U = \{z \in \mathbb{C} \mid \mathrm{dist}\,(z, \partial\Omega) < \varepsilon\},$$

for a sufficiently small value of ε. Reducing, if necessary, the value of ε, we see that by continuity of f it follows $f(U \cap \Omega) \subset \Omega \setminus U$.

Now let $\varphi : \mathbb{D} \to \Omega$ be a Riemann mapping and let $\delta > 0$. Consider the boundary of a smaller disk $\partial\mathbb{D}_{1-\delta} \subset \mathbb{D}$, which is a Jordan curve, and let $\gamma = \varphi(\mathbb{D}_{1-\delta}) \subset \Omega$ be its image in

Ω. Since φ is a conformal map, γ is also a Jordan curve in Ω, and for a small enough δ it is contained in $U \cap \Omega$ (see Figure 3.7). Therefore it follows

$$f(\gamma) \subset f(U \cap \Omega) \subset \Omega \setminus U \subset \operatorname{int}(\gamma).$$

Furthermore $\operatorname{int}(\gamma)$ contains the same number of fixed points and poles (m) of f as there are in Ω, and since γ is contained in $U \cap \Omega$ it has no fixed points nor poles in γ.

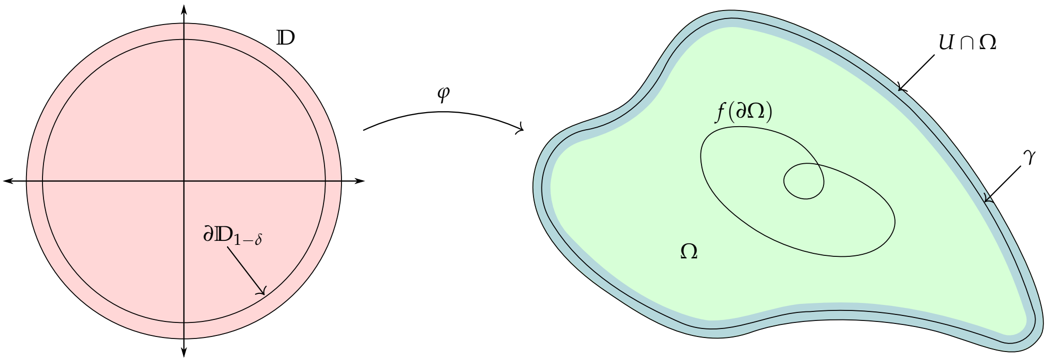


Figure 3.7: Layout of the proof of Theorem 3.20. $\mathbb{D}$ is in red, Ω in green and $U \cap \Omega$ in blue.

Now set $\sigma(t) := f(\gamma(t))$ and let $Q = \sigma(0) = \sigma(1)$. Since $\sigma = f(\gamma) \subset \operatorname{int}(\gamma)$, clearly $\gamma \cap \sigma = \varnothing$ and therefore we are under the assumptions of Lemma 3.21. Therefore we have

$$\operatorname{wind}(\sigma(t) - \gamma(t), 0) = \operatorname{wind}(\gamma, Q) + \operatorname{wind}(\sigma, z_0)$$

for any $z_0 \in \gamma$. But $\operatorname{wind}(\gamma, Q) = 1$ because γ is a Jordan curve, and $\operatorname{wind}(\sigma, z_0) = 0$ for all $z_0 \in \gamma$ because $\sigma \subset \operatorname{int}(\gamma)$, hence the left side of the previous equation is exactly 1. Notice we are also under the hypotheses of Corollary 3.22, which together with the previous result states that

$$1 = \operatorname{wind}(f(\gamma(t)) - \gamma(t), 0) = \#(\operatorname{Fix}(f) \cap \operatorname{int}(\gamma)) - \#(P(f) \cap \operatorname{int}(\gamma)),$$

where $\operatorname{Fix}(f)$ and $P(f)$ are the sets of fixed points and poles of f, respectively. Finally we obtain

$$\#(\operatorname{Fix}(f) \cap \operatorname{int}(\gamma)) = \#(P(f) \cap \operatorname{int}(\gamma)) + 1 = m + 1,$$

which concludes the proof. □

Lemma 3.23. *Let $f : \Omega \to \mathbb{C}$ be a meromorphic map and z_0 a fixed point of multiplicity $m \geq 2$. Then z_0 is a neutral or indifferent fixed point of f with multiplier 1.*

Proof. If f has a fixed point z_0 with multiplicity $m \geq 2$ there exists a neighbourhood of z_0 and a meromorphic map g in that neighbourhood such that

$$(z - z_0)^m \cdot g(z) = f(z) - z,$$

with $g(z_0) \neq 0$. Then, by differentiating both sides, we get

$$m \cdot (z - z_0)^{m-1} \cdot g(z) + (z - z_0)^m \cdot g'(z) = f'(z) - 1,$$

which evaluated at z_0, since $m \geq 2$, is simply

$$0 = f'(z_0) - 1 \Rightarrow f'(z_0) = 1,$$

which concludes that z_0 is a neutral or indifferent fixed point of f with multiplier 1. $\square$

Chapter 4

Newton's method

The purpose of this chapter is to formally define Newton's method in the complex plane and prove the two main results. First, we explore some basic facts about Newton's method in the Riemann sphere, and then we state and prove Theorem A and Theorem B. The chapter concludes with a brief overview of the relaxed Newton's method.

Back in the seventeen century, Isaac Newton used a simple technique to approximate the roots of a given polynomial. He did so by supposing an initial solution $\zeta^{(1)}$ and substituting into the polynomial $\zeta^{(1)} + \varepsilon$. Then he solved for ε the linear part of the equation, and considered a better solution $\zeta^{(2)} = \zeta^{(1)} + \varepsilon$. By iterating this numerical process he was able to find very precise solutions to complicated equations. This method was later refined by Joseph Raphson by using the derivative, obtaining the well known Newton-Raphson method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This method is used to compute the solutions of the equation $f(x) = 0$, and since it is an iterative technique it can be analyzed as the dynamical system generated by the iterates of the map

$$N_f = x - \frac{f(x)}{f'(x)},$$

which from now on will be referred as the *Newton map* of f.

4.1 Basic facts about Newton's method in the Riemann sphere

Proposition 4.1. *Consider a polynomial P in $\widehat{\mathbb{C}}$ with $d = \deg P \geq 2$ and let N_P be its Newton map. Then,*

(a) every fixed point of N_P is either a root of P or ∞.

(b) if ζ is a root of P, then ζ is an attracting fixed point of N_P. Moreover, if its multiplicity as a root of P is one, it is superattracting.

(c) $z_\infty = \infty$ is the only repelling fixed point of N_P.

Proof. To find all fixed points of $N_P(z) = z - \frac{P(z)}{P'(z)}$, we have to solve the following equation:

$$z - \frac{P(z)}{P'(z)} = z \Rightarrow \frac{P(z)}{P'(z)} = 0.$$

So any fixed point of N_P must be either a root of P or ∞. To see that any root of P is indeed an attracting fixed point of N_P, consider $P(z) = (z-\zeta)^m Q(z)$, where $m \geq 1$ and $Q(\zeta) \neq 0$. Now, $P'(z) = m(z-\zeta)^{m-1}Q(z) + (z-\zeta)^m Q'(z)$, and the Newton's function is

$$N_P(z) = z - \frac{(z-\zeta)Q(z)}{mQ(z) + (z-\zeta)Q'(z)}$$

which fixes the point ζ since the denominator would be $mQ(\zeta) \neq 0$. To deduce which type of fixed point it is we compute the derivative:

$$N_P'(z) = 1 - \frac{mQ(z)^2 + (z-\zeta)^2\left(Q'(z)^2 - Q(z)Q''(z)\right)}{\left(mQ(z) + (z-\zeta)Q'(z)\right)^2},$$

and finally

$$N_P'(\zeta) = 1 - \frac{mQ(\zeta)^2}{(mQ(\zeta))^2} = 1 - \frac{1}{m} < 1, \forall m \geq 1,$$

which proves that all roots of P are attracting fixed points of N_P. Particularly, if the root has multiplicity one, then is a superattracting fixed point.

Consider now a neighbourhood of ∞ (U) and a neighbourhood of 0 (V), and the following map: $\begin{aligned} h : U &\to V \\ z &\mapsto 1/z \end{aligned}$. Then, $\widetilde{N}_P(z) = \frac{1}{N_P(1/z)}$ is h-conjugate to $N_P(z)$:

$$\begin{tikzcd} U \arrow[r, "N_P"] \arrow[d, "h"'] & U \arrow[d, "h"] \\ V \arrow[r, "\widetilde{N}_P"] & V \end{tikzcd}$$

Simplifying the expression for $\widetilde{N}_P$, we get

$$\widetilde{N}_P(z) = \frac{zP'(1/z)}{P'(1/z) - zP(1/z)}, \tag{4.1}$$

where

$$P(z) = \sum_{i=0}^{d} a_i z^i \Rightarrow P(1/z) = \frac{\sum_{i=0}^{d} a_i z^{d-i}}{z^d}$$
$$P'(z) = \sum_{i=0}^{d} i \cdot a_i z^{i-1} \Rightarrow P'(1/z) = \frac{\sum_{i=0}^{d} i \cdot a_i z^{d-i+1}}{z^d}.$$

Substituting in 4.1, we have

$$\widetilde{N}_P(z) = \frac{zP'(1/z)}{P'(1/z) - zP(1/z)} = \frac{\sum_{i=0}^{d} i \cdot a_i z^{d-i+2}}{\sum_{i=0}^{d} i \cdot a_i z^{d-i+1} - \sum_{i=0}^{d} a_i z^{d-i+1}} = \frac{\sum_{i=0}^{d} i \cdot a_i z^{d-i+1}}{\sum_{i=0}^{d} (i-1) a_i z^{d-i}}$$

and we can see that $\widetilde{N}_P(0) = \frac{0}{(d-1)a_d} = 0$, meaning that $N_P(\infty) = \infty$, so ∞ is a fixed point. We have to check its nature, so differentiating equation 4.1 and simplifying, we obtain

$$\widetilde{N}'_P(z) = \frac{P(1/z)P''(1/z)}{(P'(1/z) - zP(1/z))^2},$$

and analogously as the former case, we get

$$\widetilde{N}'_P(0) = \frac{d}{d-1} > 1 \text{ , } \forall d \geq 2.$$

We've proven that $\widetilde{N}'_P(0) = N'_P(\infty) > 1$, which concludes that ∞ is a repelling fixed point of N_P. $\square$

Remark 4.2. Given a polynomial P in $\widehat{\mathbb{C}}$ with $d = \deg P \geq 2$, the critical points of its Newton map N_P can only be either roots or inflection points of the polynomial P, since

$$N_P(z) = z - \frac{P(z)}{P'(z)} \Rightarrow N'_P(z) = z\frac{P(z)P''(z)}{(P'(z))^2}.$$

Therefore, as seen before, the maximum number of critical points is indeed $2d - 2$.

Example 4.3. Consider the polynomial $P(z) = z^2 + c$, where $c \in \mathbb{C}$. Its Newton map is

$$N(z) = \frac{z^2 - c}{2z},$$

and the only critical points are the roots of P, $\{\pm\sqrt{-c}\}$, since P has no inflection points. In this particular case, the Julia set divides the complex plane in two half-planes, each one being the immediate basin of attraction of a root of P. So it is easy to see that the Julia set is the bisector line of the two roots, i.e., the points in the complex plane that are equally distant from both roots. The Fatou set has only two components, which are the aforementioned half-planes. This behaviour can be easily deduced from the fact that $N(z)$ is conformally conjugate to z^2 by the map

$$g(z) = \frac{z - \sqrt{-c}}{z + \sqrt{-c}},$$

which maps one fixed point to 0 and the other to ∞. The dynamical plane of this case can be seen in Figure 4.1 (a).

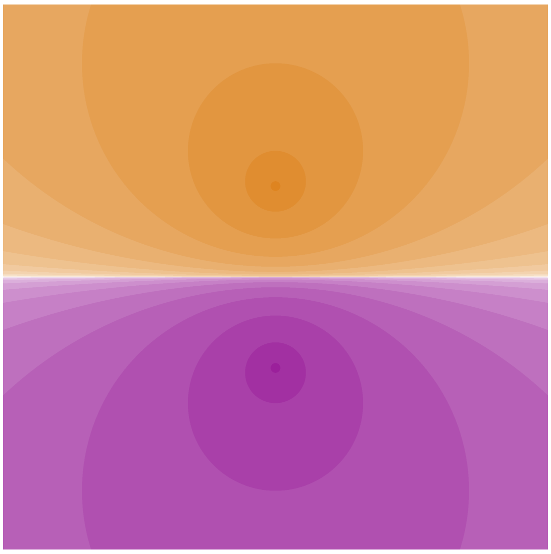
Example 4.4. Consider the polynomial $P(z) = z^n + c$, where $c \in \mathbb{C}$ and $n \geq 2$. Its Newton map is

$$N(z) = \frac{(n-1)z^n - c}{nz^{n-1}},$$

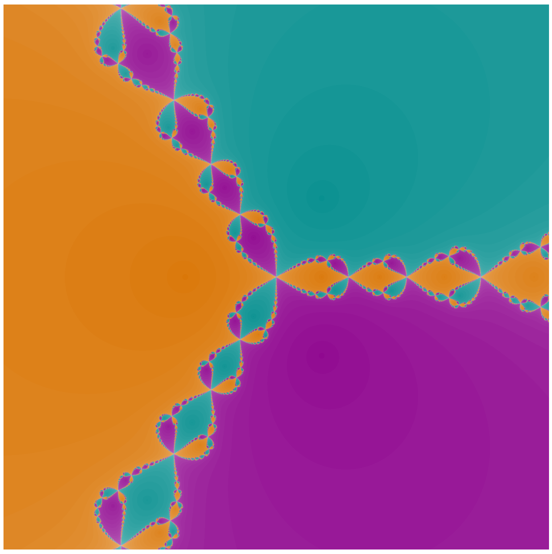
and the critical points are the solutions of the equation

$$N'(z) = \frac{n-1}{n} \cdot \frac{z^n + c}{z^n} = 0,$$

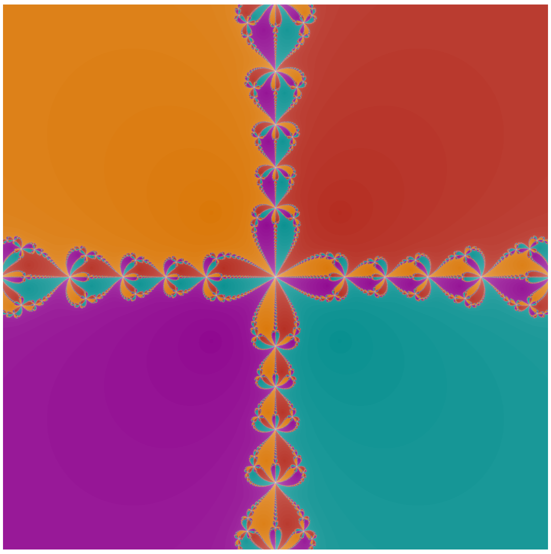
which are only the roots of P, since the polynomial P has no inflection points for any $n \geq 2$ and $c \in \mathbb{C}$. In the general case, when $n \geq 3$, the Fatou and Julia sets are much more intricate, as can be seen in Figure 4.1.



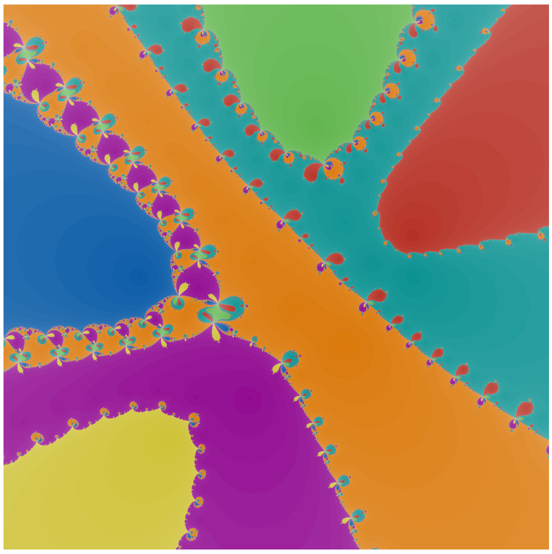
(a) Newton map of $P(z) = z^2 + 1$.



(b) Newton map of $P(z) = z^3 + 1$.



(c) Newton map of $P(z) = z^4 + 1$.



(d) Newton map of a polynomial of degree 7.

Figure 4.1: Different dynamical planes for different Newton maps. The Julia set in subfigure (a) splits the complex plane in two. Newton maps of higher degree polynomials have a much more intricate Julia set, as it can be seen in (b), (c) and (d). We can distinguish 2, 3, 4 and 7 different basins of attraction for each one of the Newton maps in (a), (b), (c) and (d), respectively, one for each root of the polynomial.

Proposition 4.5. *Consider a polynomial P in $\widehat{\mathbb{C}}$ with $d = \deg P \geq 2$. Then $z_\infty = \infty$ is the only fixed point of N_P in the Julia set.*

Proof. We have already seen in Theorem 2.44 (6) that all attracting fixed points of N_P are in $\mathcal{F}(N_P)$, and by Proposition 4.1 the only fixed point of N_P which is not attracting is $z_\infty = \infty$ (in fact, is repelling). We have also seen in Theorem 2.44 (7) that all repelling fixed points are in $\mathcal{J}(N_P)$, so we conclude that the only fixed point of N_P which is in $J(R)$ is $z_\infty = \infty$. $\square$

A key result to prove the simple connectivity of the Fatou components of a Newton map is that they cannot have Herman rings.

Proposition 4.6. *A Newton map N has no invariant Herman rings.*

Proof. Suppose N has an invariant Herman ring, U. Then by definition U is conformally equivalent to an annulus, which has infinitely many disjoint simple closed curves which are invariant under N. Let γ be one of these simple closed invariant curves, and let Ω be the open domain bounded by γ. Clearly $\Omega \cap \mathcal{J}(N) \neq \varnothing$, and notice that $N(\Omega \cap U) = \Omega \cap U$, but there must be points in Ω which are mapped outside Ω. Otherwise they would omit infinitely many points and, by Montel's Theorem, $\{N^n\}_n$ would form a normal family in Ω, contradicting the fact that $\Omega \cap \mathcal{J}(N) \neq \varnothing$.

Therefore we are under the assumptions of Lemma 3.17. $\Omega \subset \mathbb{C}$ is a simply connected bounded domain with locally connected boundary, since it is the domain bounded by a Jordan curve. Furthermore, N is meromorphic in $\mathbb{C}$, and since $N(\partial\Omega) = \partial\Omega$ we have that $\Omega = D$ is the connected component of $\widehat{\mathbb{C}} \setminus N(\partial\Omega)$. Finally, since $N(\Omega \cap U) = \Omega \cap U$ there is $z \in \Omega \cap U \subset \Omega$ with $N(z) \in \Omega \cap U \subset \Omega$, and N cannot have fixed points in $\partial\Omega$ because $N_{|U}$ is conformally equivalent to an irrational rotation. Hence by Lemma 3.17 we conclude that N has a weakly repelling fixed point in a bounded component Ω, which is a contradiction since the only weakly repelling fixed point of a Newton map is infinity. □

4.2 Theorem B: on the unboundedness of the immediate basins of attraction of N_P

Theorem 4.7. (On the unboundedness of the immediate basins of attraction of N_P) *Let P be a polynomial, $N = z - \frac{P(z)}{P'(z)}$ its Newton map, α a root of P and A the immediate basin of attraction of α. Then, A is unbounded.*

Proof. Suppose A is bounded, we will show that a fixed point of N is contained in $\partial A \subset \mathcal{J}(N)$, which contradicts Proposition 4.5 since the only fixed point of N in the Julia set is $z_\infty = \infty$.

To do so, consider $z_0, z_1 \in A$ such that $N(z_1) = z_0$, and let γ_0 be a simple curve in A joining z_0 and z_1 such that $\gamma_0 \cap P(N) = \varnothing$. Since $P(N) \setminus \{\alpha\}$ is a discrete set by Proposition 2.40, there are two open simply connected sets U_0, U_0' in A such that $\overline{U_0'} \subset U_0$, $\gamma_0 \subset U_0'$ and $U_0 \cap P(N) = \varnothing$. Since there are no critical values of N^n, $\forall n \geq 1$ in U_0, by Corollary 2.39 there are branches of N^{-n}, $\varphi_n : U_0 \to \mathbb{C}$, defined in U_0 such that $\varphi_1(z_0) = z_1$, $z_2 := \varphi_1(z_1)$, $\varphi_2(z_0) = z_2$, $\ldots$. These branches are well defined, and by taking the image of U_0, U_0' and γ_0 we can define a curve joining $z_n := \varphi_n(z_0)$ and z_{n+1} and two open neighbourhoods of theirs:

$$\gamma_n := \varphi_n(\gamma_0), \qquad U_n := \varphi_n(U_0), \qquad U_n' := \varphi_n(U_0'),$$

with $\overline{U_n'} \subset U_n$.

Note that $\forall n \geq 0$, $\gamma_n \subseteq A$: indeed, by definition $\gamma_0 \subseteq A$, and suppose that $\gamma_n \subseteq A$ but $\gamma_{n+1} \nsubseteq A$. Then, there would be a point in γ_{n+1} which would be in the Julia set, and that cannot happen because $N(\gamma_{n+1}) = \gamma_n \subset \mathcal{F}(N)$. Since both the Fatou and Julia set are completely invariant, this is a contradiction and therefore $\gamma_{n+1} \subseteq A$, completing the induction argument.

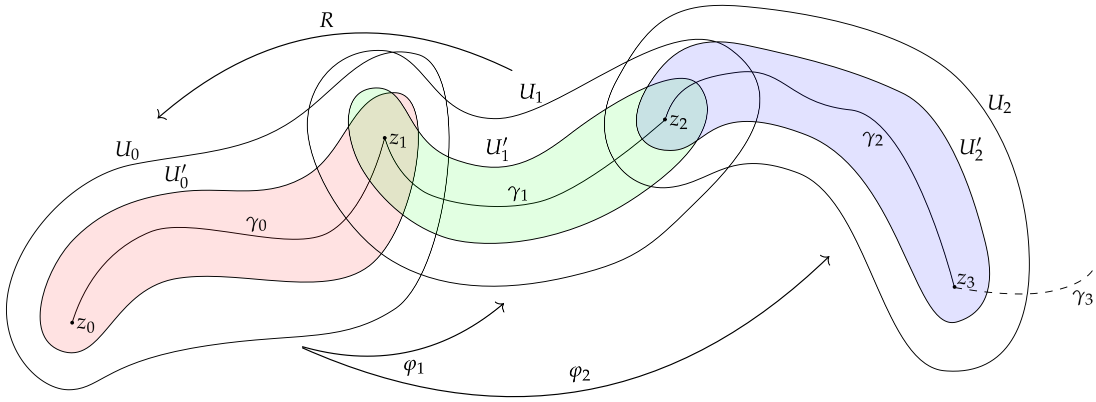


Figure 4.2: Graphic representation of the curves and sets constructed. To ease visual interpretation the sets U_0', U_1' and U_2' have been colored red, green and blue respectively.

Now, since A is bounded, $\overline{A}$ is compact, and therefore by the Bolzano-Weierstrass Theorem $\{z_n\}$ must have an accumulation point $z_{n_k} \to z_\infty$.

Consider the sets U_{n_k} which contain z_{n_k} and z_{n_k+1}, by construction. Then only two outcomes are possible:

Case (a): $\operatorname{diam} U'_{n_k} \to 0$.

In this case, applying the triangle inequality,

$$|z_{n_k+1} - z_\infty| \leq \underset{\substack{\downarrow \\ 0}}{|z_{n_k+1} - z_{n_k}|} + \underset{\substack{\downarrow \\ 0}}{|z_{n_k} - z_\infty|} \to 0 \Rightarrow z_{n_k+1} \to z_\infty,$$

where the first term tends to zero since $\operatorname{diam} \gamma_n \to 0$. Then, we have

$$N(z_\infty) = N(\lim z_{n_k+1}) = \lim N(z_{n_k+1}) = \lim z_{n_k} = z_\infty,$$

which concludes that z_∞ is a fixed point. Since the only fixed point in A is α, then $z_\infty \in \partial A \subset \mathcal{J}(N)$, which proves the result.

Case (b): $\operatorname{diam} U'_{n_k} \not\to 0$.

This means that $\exists \varepsilon > 0$ such that $\operatorname{diam} U'_{n_{k_j}} > \varepsilon$, for some subsequence $n_{k_j} \to \infty$.

Consider now the consequence of the Koebe's distortion Theorem exposed in Theorem 1.14. Since $\overline{U_0'} \subset U_0$ and the map φ_n is conformal in U_0, there's a constant $0 < k < 1$ independent of φ_n (and thus, of n) such that

$$\mathbb{D}(z_n,\ k \cdot \operatorname{diam} U'_n) \subset U'_n, \quad \forall n \geq 0.$$

Therefore, $U'_{n_{k_j}}$ contains a disk of radius $\varepsilon' = k\varepsilon$, $\forall j \geq 0$. This means that, for all j large enough, there exists $\varepsilon'' > 0$ with $\varepsilon'' < \varepsilon'$ such that

$$\mathbb{D}(z_\infty, \varepsilon'') \subset \bigcap_j \varphi_{n_{k_j}}(U_0') = \bigcap_j U'_{n_{k_j}}.$$

Then, $N^{n_{k_j}}(\mathbb{D}(z_\infty, \varepsilon'')) \subset U'_0$, meaning that there is a subsequence which remains bounded. From Lemma 2.47 we know that such a subsequence can only exist if $z_\infty \notin \mathcal{J}(N)$, meaning that $z_\infty \in \mathcal{F}(N)$.

On the other hand, z_∞ is an accumulation point of a sequence of backward iterates, and by Corollary 2.30 we know it must lay in the boundary, $z_\infty \in \partial A \subset \mathcal{J}(N)$. Hence we have seen that the accumulation point z_∞ is in both the Fatou and Julia sets, which by definition are disjoints sets. This contradiction proves that (a) is always true, and therefore the proof is finished. □

4.3 Theorem A: on the simple connectivity of all Fatou components of N_P

Having seen all previous results we are now able to proof our second main theorem. To do so, it is important to keep in mind that Newton maps have no finite weakly repelling fixed points, since all finite fixed points are roots of the polynomial and they are attracting. It is also worth noting that, since the Julia set is compact in the Riemann sphere, the connectivity of the Julia set is equivalent to the simple connectivity of all Fatou components.

Theorem 4.8. (On the simple connectivity of all Fatou components of N_P**)** *Let P be a polynomial and $N = z - \frac{P(z)}{P'(z)}$ its Newton map. Then all Fatou components of N are simply connected.*

Proof. We are going to split the proof in two parts. First we are going to focus on the invariant Fatou components, that is, the periodic components with period $p = 1$. Then we will prove the theorem for preperiodic or periodic ($p > 1$) components.

- *Invariant Fatou components:*

According to the Classification of Fatou components (Theorem 2.52), if U is an invariant Fatou component of N then it is an immediate basin of attraction of an attracting fixed point. Indeed, Proposition 4.6 rules out the possibility of U being a Herman ring, and Newton maps have no neutral or indifferent fixed points as we have seen in Proposition 4.1.

Assume U is multiply connected. Then by Lemma 3.14 there exists an $n \geq 0$ such that there is a bounded component of $\widehat{\mathbb{C}} \setminus N^n(U)$ which contains a pole p of N. Since U is invariant, the pole p is contained in a bounded component of $\widehat{\mathbb{C}} \setminus U$, which means there is a simple closed curve $\gamma' \subset U$ such that $p \in \operatorname{int}(\gamma')$. Now consider the following set

$$\Gamma' := \bigcup_{n \geq 0} N^n(\gamma') \subset U.$$

It is clear that Γ' is forward invariant, $N(\Gamma') \subset \Gamma'$:

$$N(\Gamma') = \bigcup_{n \geq 0} N^{n+1}(\gamma') = \bigcup_{n \geq 1} N^n(\gamma') \subset \bigcup_{n \geq 0} N^n(\gamma') = \Gamma'.$$

Furthermore, since $p \notin \gamma'$ and $\gamma' \subset U$, the iterates of γ' under N will be in the Fatou set, and p is in the Julia set, hence $p \notin \Gamma'$. Now, by Lemma 2.31 we know there is a simply connected absorbing domain in U. This means that eventually the image of γ' under N^n will not surround the pole p, hence there exists $n_0 > 0$ such that $p \in K(N^{n_0}(\Gamma'))$ but $p \notin K(N^n(\Gamma'))$ for all $n > n_0$. Then set

$$\Gamma := N^{n_0}(\Gamma') = \bigcup_{n \geq n_0} N^n(\gamma').$$

Let Ω' be the open connected component of $\mathbb{C} \setminus \overline{\Gamma}$ containing p, and set

$$\Omega := \bigcup \{K(\sigma) \mid \sigma \text{ is a closed curve in } \Omega'\}.$$

By definition, since Ω' is a connected component of $\mathbb{C} \setminus \overline{\Gamma}$, Ω is a bounded simply connected open set which contains p. Moreover,

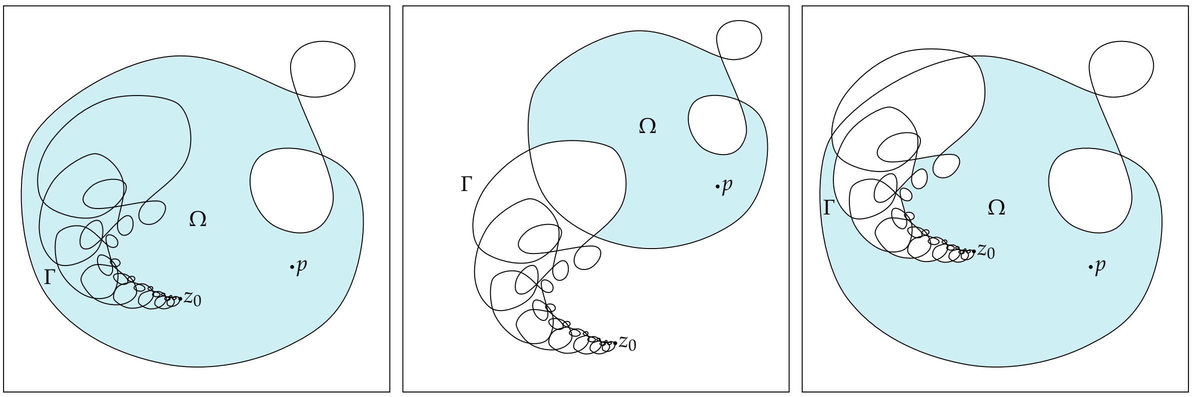
$$\partial\Omega \subset \partial\Omega' \subset \overline{\Gamma} \subset U.$$

Since Γ is also forward invariant, only one of the following can happen:

$$N(\partial\Omega) \subset \Omega \quad \text{or} \quad N(\partial\Omega) \cap \Omega = \varnothing,$$

but since no iterate of Γ can surround $p \in \Omega$, if $N(\partial\Omega) \cap \Omega = \varnothing$ then the only possibility is that $\Omega \subset \operatorname{ext}(N(\partial\Omega))$. Therefore the two cases we have to consider are the following:

$$N(\partial\Omega) \subset \Omega \quad \text{or} \quad \Omega \subset \operatorname{ext}(N(\partial\Omega)).$$



(a) (b.1) (b.2)

Figure 4.3: Possible setups in the first part of the proof of 4.8. The blue area is Ω, while p is the pole and z_0 an attracting fixed point of N.

Case (a): $N(\partial\Omega) \subset \Omega$ (see Figure 4.3 (a)).

Observe that in this case the assumptions of Theorem 3.20 are fulfilled. Indeed, $\Omega \subset \mathbb{C}$ is an open simply connected bounded domain, N is meromorphic in $\mathbb{C}$ and $N(\partial\Omega) \subset \Omega$. Furthermore Ω contains the pole p, so this means by Theorem 3.20 there are at least 2 fixed points of N in Ω counted with multiplicity. According to Lemma 3.23 any fixed point with multiplicity greater than one must be neutral, but a Newton map does not have neutral fixed points. Therefore N has at least two different fixed points in Ω. One of them may be the attracting fixed point in the immediate basin of attraction U, but the other must belong to another immediate basin of attraction U'. Otherwise it could only be infinity, but that would be a contradiction because Ω is bounded. So there must be another immediate basin of attraction U' contained in Ω, because if not its boundary, contained in $\mathcal{J}(N)$, would intersect with $\partial\Omega \subset U \subset \mathcal{F}(N)$, which is a contradiction. Hence $U' \subset \Omega$, thus U' is bounded, but that contradicts 4.7.

Case (b): $\Omega \subset \text{ext}\,(N(\partial\Omega))$.

First suppose $\overline{\Omega} \subset \text{ext}\,(N(\partial\Omega))$, so we are under the assumptions of Corollary 3.18. Indeed, if $X = \partial\Omega \subset U$, there are no poles of N in $\partial\Omega$, but $p \in K(\partial\Omega) = \overline{\Omega}$. Furthermore, $\overline{\Omega} = K(\partial\Omega) \subset \text{ext}\,(N(\partial\Omega))$. Hence N has a weakly repelling fixed point in Ω, which is a contradiction.

Now suppose the remaining scenario, this is, if $\Omega \subset \text{ext}\,(N(\partial\Omega))$ but $N(\partial\Omega) \cap \partial\Omega \neq \varnothing$. We must distinguish two separate cases.

Suppose N has no fixed points in $\partial\Omega$ (see Figure 4.3 (b.1)). Then since $\partial\Omega$ is the finite union of iterates of γ', and $\bigcup_{n=n_0}^{m} N^n(\gamma')$ is locally connected, by Torhorst Theorem (3.15) any connected component of the complement has locally connected boundary, meaning that $\partial\Omega$ is locally connected. We are thereby under the hypotheses of Lemma 3.17, with $D = \text{ext}\,(N(\partial\Omega))$ being the component of $\widehat{\mathbb{C}} \setminus N(\partial\Omega)$ containing infinity. Indeed, Ω is a simply connected bounded domain with locally connected boundary, $\Omega \subsetneq \text{ext}\,(N(\partial\Omega))$, N has no fixed points in $\partial\Omega$ and $p \in \Omega$ with $N(p) = \infty \in D$. Therefore there is a weakly repelling fixed point of N in Ω, which is a contradiction.

The only case left is when $\Omega \subset \text{ext}\,(N(\partial\Omega))$ and $\partial\Omega$ contains a fixed point z_0, particularly the attracting fixed point of the immediate basin of attraction U (see Figure 4.3 (b.2)). Note the z_0 is the only fixed point in $\partial\Omega \subset U$. Then let Δ be a small topological disk centred at z_0 such that $N(\overline{\Delta}) \subset \Delta$, which is possible since U has a simply connected absorbing domain containing z_0. Now let $\widetilde{\Omega} := \Omega \setminus \Delta$, which by construction is simply connected and has a boundary $\partial\widetilde{\Omega}$ which is locally connected. Indeed, iterates of Γ must eventually enter Δ, and therefore $\partial\widetilde{\Omega}$ is the finite union of iterates of γ' and the partial boundary of a topological disk. Finally, since $z_0 \notin \partial\widetilde{\Omega}$, we are again under assumptions of Lemma 3.17, with $D = \text{ext}\,(N(\partial\widetilde{\Omega}))$. Hence there must be a weakly repelling fixed point in $\widetilde{\Omega}$, which is impossible.

In all cases considered we reach a contradiction. Therefore the invariant Fatou component U is not multiply connected, which ends the first part of the proof.

- *Preperiodic and periodic Fatou components of period $p > 1$:*

Let U be either a periodic Fatou component of minimal period $p > 1$ or a preperiodic component, and assume U is multiply connected. Again, by Lemma 3.14 there exists an $n \geq 0$ such that there is a bounded component of $\widehat{\mathbb{C}} \setminus N^n(U)$ which contains a pole p of N, hence there is a simple closed curve $\gamma \subset N^n(U)$ surrounding p. Let $V = N^n(U)$, and Ω the bounded connected component of $\mathbb{C} \setminus \gamma$.

Note that, since $\gamma \subset V$ surrounds $p \notin \mathcal{F}(N)$, V is multiply connected, and thus it cannot be invariant by the first part of the proof. Therefore $N(\gamma)$ is in a different Fatou component than γ, i.e., $N(\gamma) \cap \gamma = \varnothing$. Then there are three different possibilities:

$$\overline{\Omega} \subset \operatorname{ext}(N(\gamma)) \quad \text{or} \quad N(\gamma) \subset \Omega \quad \text{or} \quad \gamma \subset K(N(\gamma)).$$

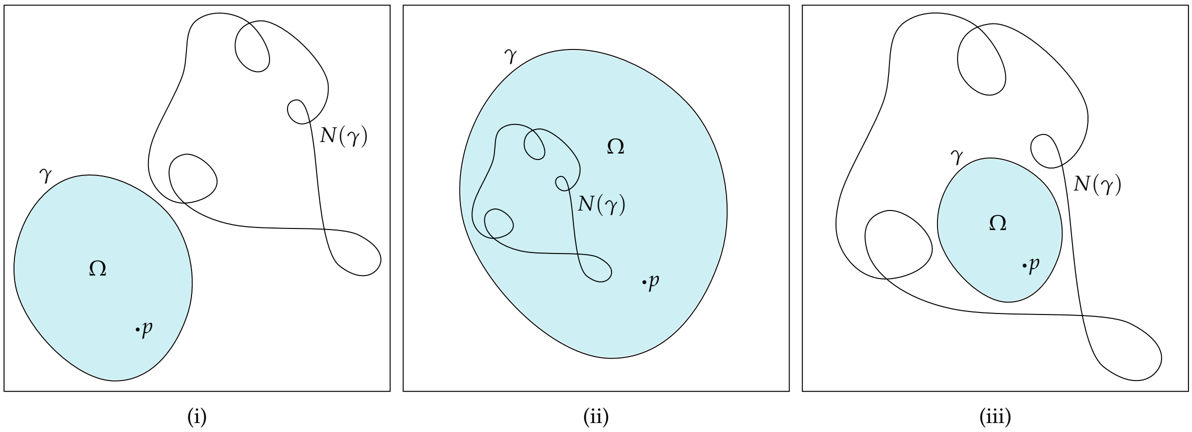


Figure 4.4: Possible setups in the second part of the proof of 4.8. The blue area is Ω, while p is the pole of N.

Case (i): $\overline{\Omega} \subset \operatorname{ext}(N(\gamma))$ (see Figure 4.4 (i)).

In the first case we are under the assumptions of Corollary 3.18 with $X = \gamma$. Indeed, $\gamma \subset \mathcal{F}(N) \subset \mathbb{C}$ is a continuum, N is meromorphic in $\mathbb{C}$ and has no poles in γ since all poles lay in the Julia set. Furthermore, $p \in \Omega \subset \overline{\Omega} = K(\gamma) \subset \operatorname{ext}(N(\gamma))$, hence Corollary 3.18 provides a weakly repelling fixed point of N in Ω, which is a contradiction.

Case (ii): $N(\gamma) \subset \Omega$ (see Figure 4.4 (ii)).

In the second case we are under the hypotheses of Theorem 3.20. Indeed, Ω is an open simply connected bounded domain, N is meromorphic in $\mathbb{C}$ and $N(\partial\Omega) \subset \Omega$ since $\partial\Omega = \gamma$. Seeing that $p \in \Omega$, there are at least 2 fixed points in Ω counted with multiplicity. Analogously to case (a) in the first part of the proof, these fixed points are distinct and attracting (since they are in a bounded domain, and N is a Newton map). Consequently their immediate basin of attraction are in Ω, and hence bounded, which is a contradiction of 4.7.

Case (iii): $\gamma \subset K(N(\gamma))$ (see Figure 4.4 (iii)).

Consider now the third case. It is clear that $N(\gamma) \subset N(V)$ also surrounds p, which makes $N(V)$ multiply connected and thereby not invariant. We have the same result as before, namely $N^2(\gamma) \cap N(\gamma) = \varnothing$, which leads to the following dichotomy:

$$N^2(\gamma) \subset K(N(\gamma)) \quad \text{or} \quad N^2(\gamma) \subset \widehat{\mathbb{C}} \setminus K(N(\gamma)) = \operatorname{ext}(N(\gamma)).$$

In the first scenario, since $N^2(\gamma) \cap N(\gamma) = \varnothing$, it follows that $N^2(\gamma)$ is in a bounded and simply connected component Ω' of $\mathbb{C} \setminus N(\gamma)$. Hence we are under the assumptions of Theorem 3.20: since $\partial\Omega' \subset N(\gamma)$, it follows $N(\partial\Omega') \subset N^2(\gamma) \subset \Omega'$. Therefore Ω' contains at least one attracting fixed point of N, whose immediate basin of attraction is contained entirely in Ω, because $\partial\Omega' \subset N(\gamma) \subset N(V)$ and $N(V)$ is not invariant and thereby it cannot contain any fixed point. Finally the boundedness of this immediate basin of attraction contradicts 4.7.

In the second scenario we are under the hypotheses of Corollary 3.19 with $X = \gamma$. Indeed, γ is a continuum in $\mathbb{C}$, N is meromorphic everywhere, N has no poles in $\gamma \subset \mathcal{F}(N)$, we have $\gamma \subset K(N(\gamma))$ and $N^2(\gamma) \subset \operatorname{ext}(N(\gamma))$. Hence N has a weakly repelling fixed point in the interior of $K(N(\gamma))$, which is a contradiction.

By assuming that U was multiply connected we have reached a contradiction in every possible scenario, meaning that our initial assumption was wrong and thereby U is simply connected, which ends this second part of the proof.

Finally, since N is a rational map, from the no wandering domain Theorem (Theorem 2.51) we know there are no more possibilities to check, which concludes that all Fatou components of N are simply connected. □

4.4 The relaxed Newton's method

The relaxed Newton's method is a slight variation of Newton's method, which makes use of a complex parameter β. It converges faster than Newton's method when applied to a polynomial with multiple roots, assuming that the multiplicity of the root is known. The relaxed Newton map is defined as follows.

Definition 4.9. (Relaxed Newton map) Let f be a complex function. Then, the *relaxed Newton map* is defined by

$$N_\beta(z) = z - \beta\frac{f(z)}{f'(z)},$$

where β is a complex constant. Note that we recover the former Newton map when $\beta = 1$.

Using the same reasoning that in Proposition 4.1, we can state the following.

Proposition 4.10. *Consider a polynomial P in $\widehat{\mathbb{C}}$ with $d = \deg P \geq 2$, and let N_β with $0 < \beta < 2$ be its relaxed Newton map. Then,*

(a) every fixed point of N_β is either a root of P or ∞.

(b) if ζ is a root of P, then ζ is an attracting fixed point of N_β. Moreover, if its multiplicity as a root of P is one and $\beta = 1$, it is superattracting.

(c) $z_\infty = \infty$ is the only repelling fixed point of N_β.

Proof. The proof of this result is analogous to that of Proposition 4.1. To avoid repetition, we will only state the multiplier of the fixed points, obtained mimicking the procedure aforementioned. If ζ is a root of P of multiplicity $m \geq 1$, and thus a fixed point of N_β, its multiplier is

$$N'_\beta(\zeta) = 1 - \frac{\beta}{m}.$$

By using the same conjugacy, we are able to find the multiplier of infinity,

$$N'_\beta(\infty) = \frac{d}{d - \beta}.$$

Observe that, as expected, when $\beta = 1$ we recover the multipliers computed in the proof of Proposition 4.1. It is also worth noting that, for multiple roots, the relaxed Newton's method performs better that the pure Newton's method, since we can set $\beta = m$ and then the root with multiplicity m will be a superattracting fixed point of N_β.

Now in order to assure that, for every $m \geq 1$, the root ζ is an attracting fixed point of N_β, we need

$$\left|N'_\beta(\zeta)\right| = \left|1 - \frac{\beta}{m}\right| < 1 \text{ }, \forall m \geq 1,$$

and this is fulfilled when $0 < \beta < 2$. Furthermore, in this case the multiplier of infinity is strictly bigger than one, hence infinity is a repelling fixed point of N_β for all $d \geq 2$.

□

The only properties of the Newton map N used in the proofs of Theorems 4.8 and 4.7 were the fixed points of N and its nature. Therefore, since N_β with $0 < \beta < 2$ has exactly the same properties as N (all finite fixed points are attracting and roots of P, and the only non-attracting fixed point is infinity, and it is strictly repelling), both theorems are fulfilled for the relaxed Newton's method with $0 < \beta < 2$.

Corollary 4.11. (On the unboundedness of the immediate basins of attraction of N_β) *Let P be a polynomial, $N_\beta = z - \beta\frac{P(z)}{P'(z)}$, with $0 < \beta < 2$, its relaxed Newton map, α a root of P and A the immediate basin of attraction of α. Then, A is unbounded.*

Corollary 4.12. (On the simple connectivity of all Fatou components of N_β) *Let P be a polynomial and $N_\beta = z - \beta\frac{P(z)}{P'(z)}$, with $0 < \beta < 2$, its relaxed Newton map. Then all Fatou components of N_β are simply connected.*

Conclusions

In order to develop the presented work, I had to dive deeply in the branch of mathematics known as complex dynamics. In order to succeed and thoroughly comprehend the two main results presented here, I had to study normal families, conformal representation, distortion theorems and proper maps, concepts that, although not being extremely difficult, were never presented in my undergraduate education. The notion of polynomial-like and rational-like maps were also very challenging, as was all the discussion made in Chapter three, when proving the existence of fixed points of meromorphic maps under certain hypotheses.

Despite being well known as a root-finding algorithm, Newton's method as a dynamical system was a point of view which I had never considered, and through this thesis I believe I have acquainted with a new branch of mathematics that I was once only able to appreciate through magnificent fractal images, and whose underlying concepts and reasonings were unknown to me.

Apart from their intrinsic appeal, the facts that Fatou components of Newton maps of polynomials (and more generally, of entire transcendental maps) are simply connected and that the immediate basins of attraction are unbounded yield interesting results. For instance, Hubbard *et al.* [HSS] have developed a technique that, given a polynomial, provides a seed in every immediate basin of attraction of the polynomial, which, under iteration of the Newton map, leads to every single root of the polynomial. As aforementioned, there are lots of unanswered questions in the complex dynamics realm, and a part of them will benefit from this recently proved result.

To conclude, this work has given me an insight into a previously unknown branch of mathematics, which, I must confess, has totally captivated me.

Appendix A

Background results

In this appendix we give some general and basic results used and cited throughout the project. The results presented here were introduced during my undergraduate education, unlike those in former sections of this thesis.

A.1 Complex analysis

Definition A.1. Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a complex map. Then, f is

(a) *holomorphic* at $z_0 \in U$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If this limit exists for every $z_0 \in U$, f is said to be *holomorphic* in U.

(b) *entire* if $U = \mathbb{C}$ and f is holomorphic in $\mathbb{C}$.

(c) *transcendental* if $U = \mathbb{C}$ and f is entire but not a polynomial.

(d) *meromorphic* in U if it is holomorphic on all of U except for a discrete set of points.

Example A.2. Every polynomial P is entire, and every rational map R is meromorphic in $\mathbb{C}$. In particular, the Newton map of a polynomial is meromorphic in $\mathbb{C}$.

Definition A.3. (Isolated singularities) Let $f : U \to \mathbb{C}$ be a complex map. If f is holomorphic in $\mathbb{D}(z_0, r) \setminus \{z_0\} \subset U$ for some $r > 0$, then f has an *isolated singularity* at z_0. Such a singularity is

(a) a *removable singularity* if there is a holomorphic function $g(z)$ in a neighbourhood W of z_0 such that $f(z) = g(z)$ for all $z \in W \setminus \{z_0\}$.

(b) a *pole* if, for $z \neq z_0$, f can be written in the form $f(z) = g(z)/h(z)$, where g and h are holomorphic at z_0, $g(z_0) \neq 0$ and $h(z_0) = 0$. We say the pole has order k if it is a zero of order k of h.

(c) an *essential singularity* if it is neither a removable singularity nor a pole.

Definition A.4. (Curve) A continuous complex map $\gamma : [a, b] \to \mathbb{C}$ with $a, b \in \mathbb{R}$ and $a < b$ is called a *curve*. By composing γ with the linear map $x \mapsto a + x(b - a)$ if needed, we can always take $[a, b] = [0, 1]$.

A curve γ is *simple* if $\gamma : [0, 1] \to \mathbb{C}$ is injective, i.e., if $\gamma([0, 1])$ does not intersect with itself.

A curve γ is *closed* if $\gamma(0) = \gamma(1)$, i.e., if it has no endpoints.

Definition A.5. (Jordan curve) A *Jordan curve* is a plane curve which is simple and closed, i.e., it is a non-self-intersecting curve with no endpoints that completely encloses an area.

Notation. The symbol γ will be used to refer to the curve and to its image in the plane, $\gamma([0, 1])$.

Remark A.6. Let γ and σ be two plane curves. If $\gamma(1) = \sigma(0)$, the curve $(\gamma + \sigma)(t)$ is defined as the concatenation of curves,

$$(\gamma + \sigma)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t < \frac{1}{2}, \\ \sigma(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

This must not be confused with the curve $\gamma(t) + \sigma(t)$, which is just the addition of the images of each curve for each value of t, simply an addition of complex numbers on the plane.

A consequence of Taylor's Theorem [MH, pp. 208-211] is the following.

Theorem A.7. (Isolated zeroes, [MH, p. 212]) *Let $f : U \to V$ be a non-zero holomorphic map, and denote by $Z(f)$ the zeroes of f. Then $\forall z_0 \in Z(f)$ there is exactly one value $m \geq 1$ such that $f(z) = (z - z_0)^m g(z)$, with g holomorphic and $g(z_0) \neq 0$. Consequently $Z(f)$ is a set of isolated points and therefore is finite or countable. If $\#Z(f) = \infty$, the zeroes of f must accumulate to ∂U.*

Corollary A.8. *Let $f : U \to V$ be a non-constant holomorphic map. Then for every $\omega \in f(U)$, the set $f^{-1}(\omega)$ is finite or countable, and if it is infinite then it must accumulate to ∂U.*

Proof. Apply the previous theorem to the non-constant holomorphic map $g(z) = f(z) - \omega$. □

Theorem A.9. (Open Mapping Theorem, [Con, p. 99]) *Let $f : U \to V$ be a non-constant holomorphic map. Then f is open, i.e., f maps open sets onto open sets.*

Definition A.10. (Winding numbers) Given γ a closed curve in $\mathbb{C}$, and $z_0 \notin \gamma$, the integer

$$\operatorname{wind}(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0}$$

is called the *winding number* or *index* of γ with respect to the point z_0. It is also the number of counter-clockwise turns that γ makes around z_0.

Proposition A.11. (Properties of winding numbers) *Let $\gamma \in \mathbb{C}$ be a closed curve and $z_0 \notin \gamma$. Then,*

(1) $\operatorname{wind}(\gamma, z) = \operatorname{wind}(\gamma, z_0)$ *for all z in the connected component of $\mathbb{C} \setminus \gamma$ containing z_0.*

(2) $\operatorname{wind}(\gamma, z_0) = 0$ *for all z_0 in the unbounded connected component of $\mathbb{C} \setminus \gamma$.*

Lemma A.12. *Let $\gamma : [0,1] \to \mathbb{C}$ be a plane closed curve and $z_0 \in \mathbb{C} \setminus \gamma$ a point. Then,*

$$\operatorname{wind}(\gamma - z_0, 0) = \operatorname{wind}(\gamma, z_0).$$

Proof. Using the definition of winding number, it is equivalent to show

$$\int_{\gamma - z_0} \frac{\mathrm{d}z}{z} = \int_{\gamma} \frac{\mathrm{d}z}{z - z_0}.$$

We know the primitive of those integrals is the logarithm, so we have

$$\int_{\gamma - z_0} \frac{\mathrm{d}z}{z} = \log(\gamma(1) - z_0) - \log(\gamma(0) - z_0) = \int_{\gamma} \frac{\mathrm{d}z}{z - z_0},$$

which ends the proof. □

Definition A.13. (Homotopic curves) Let $\gamma, \widetilde{\gamma} : [0,1] \to U \subset \mathbb{C}$ be two curves. We say that γ and $\widetilde{\gamma}$ are *homotopic in* U, and denote it by $\gamma \sim \widetilde{\gamma}$, if there exists a continuous map

$$\begin{array}{rccl} H: & [0,1] \times [0,1] & \longrightarrow & U \subset \mathbb{C} \\ & (t,s) & \longmapsto & H(t,s) = \gamma_s(t) \end{array}$$

such that $\gamma_0(t) = \gamma(t)$ and $\gamma_1(t) = \widetilde{\gamma}(t)$, for all $t \in [0,1]$. The map H is called *homotopy*. Particularly, we say γ is *homotopic in* U *to a point* z_0 if it can be continuously deformed to the constant curve z_0.

Theorem A.14. (Deformation Theorem, Theorem 2.3.12 in [MH]**)** *Let f be an entire function in U, and let $\gamma, \widetilde{\gamma}$ two homotopic closed C^1-curves in U. Then*

$$\int_{\gamma} f = \int_{\widetilde{\gamma}} f.$$

Although the result presented is for differentiable curves, it is also valid for continuous curves. This is because any continuous curve can be approximated by polygons.

Lemma A.15. (Invariance of the winding number under homotopies) *Let $\gamma, \widetilde{\gamma} : [0,1] \to \mathbb{C} \setminus \{z_0\}$ be two homotopic curves in $\mathbb{C} \setminus \{z_0\}$. Then,*

$$\operatorname{wind}(\gamma, z_0) = \operatorname{wind}(\widetilde{\gamma}, z_0).$$

Proof. This is a direct application of the Deformation Theorem, since $\gamma \sim \widetilde{\gamma}$ means that

$$\operatorname{wind}(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{\mathrm{d}z}{z - z_0} = \frac{1}{2\pi i} \int_{\widetilde{\gamma}} \frac{\mathrm{d}z}{z - z_0} = \operatorname{wind}(\widetilde{\gamma}, z_0).$$

□

Corollary A.16. *Let $\gamma, \widetilde{\gamma}, \sigma, \widetilde{\sigma} : [0,1] \to \mathbb{C}$ be curves such that $\gamma \cap \sigma = \varnothing$, $\gamma \sim \widetilde{\gamma}$ in $\mathbb{C} \setminus \sigma$ and $\sigma \sim \widetilde{\sigma}$ in $\mathbb{C} \setminus \gamma$. Then,*

$$\operatorname{wind}(\gamma - \sigma, 0) = \operatorname{wind}(\widetilde{\gamma} - \widetilde{\sigma}, 0).$$

Proof. Let $\Gamma_s(t) := \gamma_s(t) - \sigma_s(t)$. Then, $\Gamma(t) := \Gamma_0(t) = \gamma(t) - \sigma(t)$ and $\widetilde{\Gamma}(t) := \Gamma_1(t) = \widetilde{\gamma}(t) - \widetilde{\sigma}(t)$, therefore $\Gamma \sim \widetilde{\Gamma}$ in $\mathbb{C} \setminus \{0\}$. Since homotopies preserve the winding number, applying the previous lemma we are done. □

Theorem A.17. (Argument Principle, [Con, p. 123]) *Let $\Omega \subset \mathbb{C}$ be an open set and f a meromorphic non-constant map in every connected component of Ω. Let Z and P be the zeros and poles of f, respectively, and let $\gamma \subset \Omega$ be a closed curve such that*

(a) $\operatorname{wind}(\gamma, z) = 0$ *for all $z \notin \Omega$,*

(b) there are no zeros nor poles of f on γ, i.e., $\gamma \cap (Z \cup P) = \varnothing$.

Then,

$$\operatorname{wind}(f(\gamma), 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{z \in Z} \operatorname{wind}(\gamma, z) \cdot m(z) + \sum_{p \in P} \operatorname{wind}(\gamma, p) \cdot m(p),$$

where $m(z)$ and $m(p)$ are the multiplicities of the zeros and poles, respectively.

Theorem A.18. (Rouché's Theorem, [Con, p. 125]) *Let $U \subset \mathbb{C}$ be an open set, γ a simple closed curve in U and f, g holomorphic maps on U such that $|f(z) - g(z)| < |f(z)|$ on γ. Then f and g have the same number of zeroes inside the connected component enclosed by γ, counted with multiplicity.*

Theorem A.19. (Mapping Theorem, [MH, pp. 399-400]) *Let $f : U \to V$ be a non-constant holomorphic map, and $a \in U$ and $b \in V$ points such that $f(a) = b$. Then exist open neighbourhoods $A \subset U$ of a and $B \subset V$ of b such that every point $\omega \in B$ has the same number of preimages in A counted with multiplicity. Additionally,*

- *if $f'(a) \neq 0$ the number of preimages is 1.*
- *if $f'(a) = f''(a) = \cdots = f^{(k-1)}(a) = 0$ but $f^{(k)}(a) \neq 0$, the number of preimages is k and all are regular except a, i.e., $f'(z) \neq 0$ for all $z \in A \setminus \{a\}$.*

Proof. Since f is a non-constant holomorphic map, the zeros of $f(z) - b$ are isolated, so there is an $\varepsilon > 0$ such that $f(z) - b$ has no zeros in $|z - a| \leq \varepsilon$ other than a. Therefore, on the compact set $\gamma = \{z \in U \mid |z - a| = \varepsilon\}$ the map $f(z) - b$ is continuous and never zero, meaning that there is $\delta > 0$ such that $|f(z) - b| \geq \delta$ for every $z \in \gamma$. So if $\omega \in V$ satisfies $|\omega - b| < \delta$, then for every $z \in \gamma$ we have $f(z) - \omega \neq 0$, because otherwise we would have $f(z) = \omega$ and thus $|\omega - b| \geq \delta$. Moreover, we have

$$|(f(z) - b) - (f(z) - \omega)| = |b - \omega| < \delta \leq |f(z) - b|\,.$$

Now applying Rouché's Theorem to $g(z) = f(z) - b$ and $h(z) = f(z) - \omega$ in the simple curve γ, we know that g and h have the same number of zeros counted with multiplicity inside γ, i.e., every point $\omega \in \mathbb{D}(b, \delta)$ has the same number of preimages in $\mathbb{D}(a, \varepsilon)$ under f, counted with multiplicity.

Additionally, since f is non-constant, its derivative is not identically zero, and thus its zeroes are isolated. Suppose that $f'(a) \neq 0$. Then from continuity of the inverse function, the inverse function is holomorphic and thus f is conformal, meaning that the number of preimages is exactly one.

Suppose now that $f'(a) = f''(a) = \cdots = f^{(k-1)}(a) = 0$ but $f^{(k)}(a) \neq 0$. This means there is an holomorphic map g at a such that $f(z) = b + (z-a)^k g(z)$, with $g(a) \neq 0$. Indeed, the derivatives of f are

$$f^{(n)}(z) = (z-a)^{k-n} \left(\frac{k!}{(k-n)!} \cdot g(z) + (z-a) \cdot h_n(z) \right), \text{ for } 1 \leq n \leq k,$$

for some analytic map $h_n(z)$ in a. Therefore the first non-zero derivative at a is $f^{(k)}(a) = k! \cdot g(a) \neq 0$, and so the equation $\omega - b = (z-a)^k g(z)$ in the neighbourhood $\mathbb{D}(a, \varepsilon)$ must have k solutions, counted with multiplicity. Hence f has k preimages in $\mathbb{D}(a, \varepsilon)$. $\square$

Proposition A.20. *If $f : U \to V$ is a non-constant holomorphic and bijective map in U, the inverse $f^{-1} : V \to U$ is holomorphic in V.*

Proof. If f is non-constant and holomorphic, it is an open map. Thus, since f is bijective, its inverse $f^{-1} : V \to U$ is continuous because the inverse of the inverse (i.e. f) maps open sets onto open sets. Then, to prove that f^{-1} is holomorphic in V, we have to prove that $\forall \omega_0 \in V$ the following limit exists:

$$\lim_{\omega \to \omega_0} \frac{f^{-1}(\omega) - f^{-1}(\omega_0)}{\omega - \omega_0}.$$

If it does, it is precisely $(f^{-1})'(\omega_0)$. Since f is bijective, for every $\omega \in V$ there is a point $z \in U$ such that $f(z) = \omega$. So we have the following:

$$\lim_{\omega \to \omega_0} \frac{f^{-1}(\omega) - f^{-1}(\omega_0)}{\omega - \omega_0} = \lim_{z \to z_0} \frac{f^{-1}(f(z)) - f^{-1}(f(z_0))}{f(z) - f(z_0)} = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)^{-1} =$$
$$= \left(\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \right)^{-1} = \left(f'(z_0) \right)^{-1} = \frac{1}{f'(f^{-1}(\omega_0))} = (f^{-1})'(\omega_0),$$

thus the limit exists and therefore f^{-1} is holomorphic in V. $\square$

Theorem A.21. (Bloch's Theorem, [Con, pp. 293-295]) *Let f be a holomorphic function on a region containing the closure of the unit disk $\overline{\mathbb{D}}$ such that $f(0) = 0$ and $f'(0) = 1$. Then $f(\mathbb{D}) \supset \mathbb{D}_L$, where $L = 1/72$.*

A.2 Plane topology

Definition A.22. (Topological disk) A *topological disk* is a non-empty simply connected open set, i.e., a surface homeomorphic to a disk in a plane.

Definition A.23. (Euler characteristic) Let $U \subset \widehat{\mathbb{C}}$, and T a triangulation of U. Then the *Euler characteristic* of U is

$$\chi(U) = F - E + V,$$

where F, E and V are the number of faces, edges and vertices of the triangulation T, respectively.

Proposition A.24. **([Bea, pp. 83-85])** *Let $D \subseteq \widehat{\mathbb{C}}$. Then,*

(a) $\chi(D) = 2$ *if and only if D is the Riemann sphere $\widehat{\mathbb{C}}$.*

(b) $\chi(D) = 2 - k$ *if and only if D is $k-$connected, for $k \geq 1$.*

Definition A.25. **(Continuum)** A *continuum* is a non-empty compact connected metric space.

Lemma A.26. *Let $f : A \cup B \to \mathbb{C}$ be a complex map. Then,*

(a) $f(A \cap B) \subset f(A) \cap f(B)$,

(b) $f(A \cup B) = f(A) \cup f(B)$.

Proof.

(a) Let $x \in A \cap B$. Since $x \in A$, $f(x) \in f(A)$, and also since $x \in B$, $f(x) \in f(B)$. Therefore $f(x) \in f(A) \cap f(B)$.

(b) Let $x \in A \cup B$. If $x \in A$, then $f(x) \in f(A) \subset f(A) \cup f(B)$. On the other hand, If $x \in B$, then $f(x) \in f(B) \subset f(A) \cup f(B)$. To prove the other inclusion, let $y \in f(A) \cup f(B)$. Now, if $y \in f(A)$ then there exists $x \in A$ such that $y = f(x)$, so it follows that $x \in A \subset A \cup B$, meaning that $y = f(x) \in f(A) \subset f(A \cup B)$. Analogously for $y \in f(B)$, we obtain $y \in f(A \cup B)$, proving the second inclusion and therefore finishing the proof.

□

Proposition A.27. *Let $f : U \to V$ be a continuous map. Then, if U is connected the image $f(U)$ is also connected.*

Proof. Suppose $f(U)$ is not connected. Then there exist two open sets A, B such that $A \cap B = \varnothing$, $f(U) \subset A \cup B$ and $A \cap f(U) \neq \varnothing$ as well as $B \cap f(U) \neq \varnothing$. Note that if $x \in f^{-1}(A) \cap f^{-1}(B)$, then $f(x) \in f(f^{-1}(A) \cap f^{-1}(B)) \subset A \cap B = \varnothing$, which is a contradiction. Furthermore, if $x \in U$ then $f(x) \in f(U) \subset A \cup B$, meaning that $x \in f^{-1}(A) \cup f^{-1}(B)$. Therefore $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint open sets covering U, so U is not connected, which is a contradiction. □

Appendix B

Code and images

All the images shown in this thesis have been generated using python scripts, fully developed by the author. The code used can be found in the following GitHub repository:

`https://github.com/pedemonte96/TFG_MATHS_2020`

In the link above there can also be found all the images used throughout the project with higher resolution, as well as other interesting images not included in this work.

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